Combinatorics 1.

K1.1. Estonian Mathematical Olympiad 1999 Final Round 10th Grade 4/5.

32 stones, with pairwise different weights, and lever scales without weights are given. How to determine by 35 scaling, which stone is the heaviest and which is the second by weight?

K1.2. Russian Mathematical Olympiad 2002 IV-th (District) round 8-th form 8/8.

18 parts are arranged in a line. It is known that there are 3 consecutive details which weigh 99 g each and that all the others weigh 100 g. You are allowed to chose two groups of parts and to weigh each of these groups. Determine which parts weigh 99 g.

K1.3. Macedonian Mathematical Competition 2002 I Round IV Class

Calculate the coefficient in front of $x^n y^n$ in the binomial expansion of $(1 + x)^n(1 + y)^n(x + y)^n$.

K1.4. Auckland Mathematical Olympiad 1998: Division 2 2/5.

Some cells of an infinite square grid are coloured black and the rest are coloured white so that each rectangle consisting of 6 cells (2x3 or 3x2) contains exactly 2 black cells. How many black cells might a 9x11 rectangle contain?

K1.5. Austrian Mathematical Olympiad (Final Round, Day 1): June 2002 1/3.

An 8x8 chess board contains many rectangles and squares composed completely of whole squares on the board. These range in size from a 1x1 square to an 8x8 square. What is the sum of the areas of all such squares and rectangles? Determine the formula for the sum of all such areas for an axb chess board, from the 1x1 squares to the axb rectangle.

Combinatorics 2.

K2.1. Hellenic Math. Competition April 2002 (Final Selection Ex. for Juniors) 4/4.

We have 100 cards with two sides, „even” and the „odd”. At each card are written two successive integers, one odd integer on the „odd” side and back on the „even” side the even integer following the odd integer of the „odd” side. All integers from 1 to 200 have been used.
The student A selects (randomly) 21 cards and adding up their integers of both sides announces the sum 913.
The student B selects (randomly) from the cards left 20 cards and adding up their integers of both sides announces the sum 2400.
a) Explain why the summation of A has a mistake.
b) If the correct sum for A is 903, explain why the summation of B has a mistake.

K2.2. Japanese Mathematical Olympiad 1998 First Round

Let $a_n = 1998 \cdot 2^{n-1}$ with $1 \le n \le 100$. How many a_n are there that the leftmost digit of the decimal representation of a_n is 1?

K2.3. USSR Maths Olympiad 1989

Prove that it is not possible to pack 77 1x3x3 bricks into a box of dimension 7x9x11.

K2.4. Mathematical county competition in Croatia 2002: 4^{th} grade 3/4.

Nine students are participating in a mathematical competition. It is known that each of the given problems was solved by exactly three students, and that for every pair of students there exist exactly one problem that was solved by both of them. Determine how many problems were given at this competition.

K2.5. Estonian Mathematical Olympiad 1999 Final Round 12^{th} Grade 4/5.

Let us put pieces on some squares of 2nx2n chessboard in such a way that on every horizontal and vertical line there is an odd number of pieces. Prove that the whole number of pieces on the black squares is even.

Pigeonhole-principle 1.

K3.1. Math Olympiad in Slovenia 1998 First Round Grade Four 4/4.

A tribe has 90 warriors and every warrior has a lance, decorated by 9 red or gold-yellow feathers of paradisiacal bird. The feathers are put on the lance in a sequence and no two red feathers next to each other. Is it possible that all the lances decorated differently?

K3.2. Greek Mathematical Olympiad 2000 4/4.

For the subset $A_1, A_2, \ldots, A_{2000}$ of the set M, we have $|A_i| \geq \frac{2|M|}{3}$, i = 1, 2, … , 2000, where $|X|$ denotes the cardinality of the set X. Prove that there exists $m \in M$ which belongs to at least 1334 ones of the subsets A_i.

K3.3. Bulgarian Mathematical Olympiad 1994

Thirty-three natural numbers are given. The prime divisors of each of the numbers are among 2, 3, 5, 7, 11. Prove that the product of two of the numbers is a perfect square.

K3.4. Mathematical competition in Lithuania October 2001

The chessboard of size 6x6 is covered by 18 dominoe stones 2x1. Each stone covers exactly two fields of the chessboard. Prove that it is possible by one vertical or horizontal cut to divide the chessboard into two not necessarily equal parts without cutting any dominoe stone.

K3.5. British Maths Olympiad 2000 Round 2

a) Find a set A of ten positive integers such that no six distinct elements of A have a sum which is divisible by 6.
b) Is it possible to find such a set if „ten” is replaced by „eleven”?

Pigeonhole-principle 2.

K4.1. Math Olympiad in Slovenia 1998 First Round Grade Three 4/4.

Twelve persons took a seat at a round table but no one was sitting on the chair assigned by the host. Prove that at least two persons will be seated on the right chairs if all the persons just do an appropriate circular move round the table.

K4.2. Hellenic Mathematical Olympiad February 2002 2/4.

A student of the National Technical University was reading Higher Mathematics last summer for 37 days according to the following rules:
(1) Every day he was reading at least one hour.
(2) Every day he was reading an integer number of hours and at most twelve hours.
(3) Totally he had to read at most 60 hours.
Prove that there were some successive days during which the student read totally 13 hours.

K4.3. Russian Mathematical Olympiad 2002 IV-th (District) round 10-th form 2/8.

A convex polygon on the plane contains at least $m^2 + 1$ points with integer coordinates. Show that one can find inside the polygon $m + 1$ points with integer coordinates lying on the same line.

K4.4. Mathematical Olympiads' Correspondence Program 1996 Canada

None of the nine participants in a scientific symposium speaks more than three languages. Two of any three participants speak a common language. Show that there is a language spoken by at least three participants.

K4.5. Irish Mathematical Olympiad 1999: First Day (Time: 3 hours) 2/5.

Show that there is a positive number in the Fibonacci sequence which is divisible by 1000. (The Fibonacci sequence F_n is defined by the conditions $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. So the sequence begins 0, 1, 1, 2, 3, 5, 8, 13, …)

Pigeonhole-principle 3.

K5.1. German National Mathematics Competition 1999 First Round

1600 coconuts are distributed over 100 monkeys, allowing some monkeys to be left empty-handed. Prove that – independent from the distribution – there are always four monkeys with the same number of coconuts.

K5.2. Math Olympiad in Slovenia 2002 Final Round Grade Three 4/4.

In the cellar of a castle 7 dwarfs protect their treasure. The treasure is behind 12 doors and each door has 12 locks. All locks are different from each other. Every dwarf has the keys for some locks. Any three dwarfs together have keys for all locks. Prove that all dwarfs together have at least 333 keys.

K5.3. Mathematical Competition 1997 Lithuania

Is there at least one positive integer *n*, such that the last 1997 digits of the number $1997^n - 1$ are zeros?

K5.4. Mathematical Competition 1990 New Zealand

Given 101 rectangles each of whose sides has integer length not exceeding 100, prove that there are three distinct rectangles A, B, C, such that A will fit inside B and B will fit inside C. (An m x n rectangle will fit inside an m' x n' rectangle if $m < m'$ and $n \leq n'$ or if $m \leq m'$ and $n < n'$.)

K5.5. Serbian Mathematical Olympiad 1999 1st form (Time: 4 hours) 2/5.

Given a set $A \subset \{1, 2, \dots, 100\}$ which contains ten elements. Prove that there exist two disjoint and nonempty subsets S and T of the set A, so that the sum of elements of set S is equal to the sum of elements of set T.

Existence and construction in combinatorics 1.

K6.1. Math Olympiad in Slovenia 2002 Final Round Grade One 4/4.

At least how many stars should be drawn in a table 4x4 such that after eliminating arbitrary two columns and arbitrary two rows there should still some stars be left ont he remaining table?

K6.2. Estonian Spring Open Contest: February 2002: Seniors 5/5.

Find the maximum number of distinct four-digit positive integers consisting only of digits 1, 2 and 3 such that any two of these numbers have equal digits in at most one position?

K6.3. Mathematical Competition 1992 New Zealand

A dress maker wants a set of six separate rods of integer length which will enable her to measure every length from 1 cm to 63 cm using various rods end to end. What lengths should she use and why?

K6.4. Mathematical Olympiads' Correspondence Program 1996 Canada

Place 32 white and 32 black checkers on a standard 8x8 checkerboard. Two checkers of different colours will be said to form a „related pair" if they are placed either in the same row or the same column. Determine the maximum and the minimum number of related pairs (over all possible arrangements of the checkers).

K6.5. Russian Mathematical Olympiad 2002 IV-th (District) round 10-th form 8/8.

What is the largest possible number of colors in which one can paint all the squares of a 10x10 checkerboard so that each of its columns and each of its rows would be painted in at most 5 different colors?

Existence and construction in combinatorics 2.

K7.1. Serbian Mathematical Olympiad 1999 3rd form (Time: 4 hours) 3/5.

Find the greatest number of equilateral crosses of the area of 5 that can be cut from the table 6x6.

K7.2. Estonian Open Contest 1999 (9th and 10th grade)

For which values of n ($n \geq 3$) is it possible to draw on a plane such a closed broken line consisting of n links that every link has exactly one point in common with every other link so that this point is an endpoint or an internal point of both links, and no point on the plane is an endpoint for more than two links?

K7.3. Bulgarian National Olympiad 1993

A regular hexagon with a side of length 1 is given. Find the greatest positive integer n for which here exist n points in the interior or on the sides of the given hexagon, such that the distance between any two of them is not less than $\sqrt{2}$.

K7.4. Irish Mathematical Olympiad 1998: Second Day (Time: 3 hours) 3/5.

Let N be the set of natural numbers (i. e., the positive integers).
a) Prove that N can be written as a union of three mutually disjoint sets such that, if $m, n \in N$ and $|m - n| = 2$ or 5, then m and n are in different sets.
b) Prove that N can be written as a union of four mutually disjoint sets such that, if $m, n \in N$ and $|m - n| = 2$, 3 or 5, then m and n are in different sets. Show, however, that it is impossible to write N as a union of three mutually disjoint sets with this property.

K7.5. German National Mathematics Competition 1999 First Round

There exist convex polyhedrons having more sides than vertices. Determine with proof the smallest number of triangular sides that such a polyhedron can have.

Algorithm 1.

K8.1. Auckland Mathematical Olympiad 1997: Division 1 5/5.

Given a number written on the blackboard in one move you can either replace it by this number squared or else by this number doubled. How to get 2^{45} exactly in 10 moves starting from?

K8.2. Mathematical competition in Lithuania October 2001

Is it possible to express $\frac{1}{2}$ by a finite sum $\frac{1}{n_1^2}+\frac{1}{n_2^2}+...+\frac{1}{n_l^2}$, where all positive integers n_1, n_2 , … , n_l are different?

K8.3. Math Olympiad in Slovenia 1998 Final Round Grade One 4/4.

In the lower left corner of the 8x8 chessboard there are 9 pawns, denoted by •. Each pawn can be moved over a neighbouring pawn and put on a field behind that pawn if that field is empty. (Two fields are neighbouring if they have a common side or a common vertex.) Is it possible to transfer all the nine pawns to the fields denoted by o using the described moves only?

o	o	o					
o	o	o					
o	o	o					
•	•	•					
•	•	•					
•	•	•					

Note:

We would like to mention the following problem.

K8.4. There are 9 counters in the bottom-left 3x3 fields of a chessboard. With a counter we may jump over one of its neighbours if the landing field is empty. (Neighbours have at least one common vertex.) If a counter has no upper neighbour we may move it to that field. Determine the minimal number of steps we need to move the 9 counters to the top-left 3x3 corner.

K8.5. Konhauser Poblemfest April 2001 Canada

a) Begin with a string of 10 A’s, B’s, and C’s, for example A B C C B A B C B A and underneath, form a new row, of length 1 shorter, as follows: between two consecutive letters that are different, you write the third letter, and between two letters that are the same, you write that same letter again. Repeat this process until you have only one letter in the new row. For example, for the string above, you will now have:

```
A B C C B A B C B A
 C A C A C C A A C
  B B B B C B A B
   B B B A A C C
    B B C A B C
     B A B C A
      C C A B
       C B C
        A A
         A
```

Prove that the letters at the corners of the resulting triangle are always either all the same or all different.

b) For which positive integers n (besides 10) is the result from part a) true for all strings of n A's, B's, and C's?

Algorithm 2.

K9.1. Russian Mathematical Olympiad 2002 IV-th (District) round 8-th form 5/8.

A four digit number is written on the blackboard. One is allowed to add 1 to any two neighboring digits if neither of them is 9, or to subtract 1 from any two neighboring digits if neither of them is 0. Can one obtain 2002 from 1234 by performing such operations arbitrarily many times?

K9.2. Mathematical Competition 1997 Lithuania

The numbers $\sqrt{2}$, 2, and $\frac{1}{\sqrt{2}}$ are written on the blackboard. We may erase any two of them writing down of them their sum and difference both divided by $\sqrt{2}$ instead. Is it possible to get the triplet 1, $\sqrt{2}$, and 1 + $\sqrt{2}$ ont he course of a procedure?

K9.3. South Africa, Potchefstroom Camp July 2001: Test 1 (Time: 4 hours 15 m.) 4/5.

A computer screens shows a 98x98 chessboard, coloured in the usual way. One can select with a mouse any rectangle with sides on the lines of the chessboard and click the mouse button: as a result, the colours in the selected rectangle switch (black becomes white, white becomes black). Find, with proof, the minimum number of mouse clicks needed to make the chessboard all one colour.

K9.4. Brazilian Mathematical Olympiad 2001 Second day 1/3.

Uma calculadora tem o número 1 na tela. Devemos efetuar 2001 operaçőes, cada uma das quais consistindo em pressionar a tecla sen ou a tecla cos. Essas operaçőes calculam respectivamente o seno e o cosseno com argumentos em radianos. Qual é o maior resultado possível depois das 2001 operaçőes?

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Initially, a calculator displays the number 1. An operation consists in pressing either key sin or key cos, which calculates respectively the sine and cosine of the arguments in radians. After performing 2001 operations, what is the greatest possible value that can be achieved?

K9.5. Iranian Mathematical Olympiad 2002 First Round 6/6. Time: 2x4 hours

Consider an infinite strip of squares. Some of the squares of this strip are occupied by coins. (A square may contains more than one coin.) In each step we may perform one of the following moves:

1) If there are some coins in two neighboring squares (indexed by n – 1, n) then we can remove one coin from both of the squares n – 1, n and place one coin on the $(n + 1)^{th}$ square.
2) We can remove two coins from n^{th} square ($n \geq 3$) and place one coin on each of the $(n + 1)^{th}$, $(n – 2)^{th}$ squares.
a) Prove that any sequence of such moves will lead to a position where there are no further moves possible.
b) Suppose that in each of the first n squares there is a single coin.. Show that no matter how we proceed, no coin can be placed on the $(n + 1)^{th}$ square.

Games for 2 persons

K10.1. Math Olympiad in Slovenia 1998 Final Round Grade Two 4/4.

Let us examine a game for two players. At the beginning a pile of at least two stones is given. The player who is in turn chooses one of the piles and divides it into two or three new piles of at least one stone each. The player who can not divide any pile any more is the looser. Who wins if they are both clever?

K10.2. Estonian Open Contest 1998

Consider two players, Mary and Peter playing the following game on 100xn ($n \geq 2$) table (the direction of 100 squares we call horizontal and the direction of n squares vertical). Mary has one white piece initially situated on the lower right square, Peter has one black piece initially on the upper right square. Player can move his (her) piece only one square horizontally or vertically to an empty square during one move. Mary moves first, later the moves will be performed alternately. Mary wins if she is able to move her piece to the upper row, Peter wins if some situation on the table repeats. Who wins using the right strategy?

K10.3. Estonian Open Contest 1998

Consider two players, Mary and Peter playing the following game on 100xn ($n \geq 2$) table (the direction of 100 squares we call horizontal and the direction of n squares vertical). Mary has one white piece initially situated on the lower right square, Peter has two black pieces both initially on the upper right corner. Mary can move her piece only one square horizontally or vertically to an empty square during one move and in one move Peter moves both of them. Mary moves first, later the moves will be performed alternately. Mary wins if she is able to move her piece to the upper row, Peter wins if Mary is not able to perform a move and the game ends with draw, if some situation on the table repeats. Who wins using the right strategy?

K10.4. Serbian Mathematical Olympiad 1999 2^nd^ form (Time: 4 hours) 3/5.

The expression $f(x) = x^3 + ax^2 + bx + c$ is written on the board. Two students play the following game: the first player replaces one of the a, b and c with some real number. After that, the second player does the same with one of remaining numbers. At the end, the first player replaces last parameter with a real number. If the arising polinomial has no positive roots, the first student is the winner, otherwise the second student. Find out who can win and how.

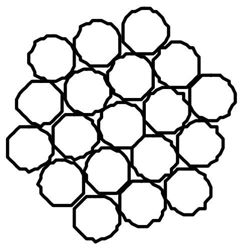
K10.5. South Africa, Stellenbosch Camp December 1998: Test 4 (Time: 3 hours) 1/3.

Let the polynomial $f(x) = x^{10} + *x^9 + *x^8 + \ldots + *x + 1$ be given, when the nine starred coefficients are to be filled in by two players playing alternately until no stars remain. The first player wins if all the zeros of the polynomial are nonreal; the second player wins otherwise. Show that there is always a winning strategy for the second player.

Recursion in combinatorics 1.

K11.1. Australian Mathematical Olympiad 1996

A number of tubes are bundled together into a hexagonal form, as shown on the figure. The number of tubes in the bundle can be 1, 7, 19(as shown), 37, 61, 91, … If this sequence is continued, it will be noticed that the total number is often a number ending in 69. What is the 69^{th} number in the sequence which ends in 69?



K11.2. British Mathematical Olympiad December 2001: Round 1 (Time: 3.5 hours) 4/5.

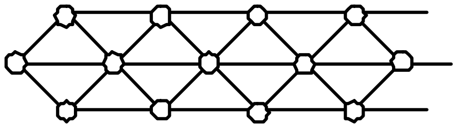
Twelve people are seated around a circular table. In how many ways can six pairs of people engage in handshakes so that no arms cross?
(Nobody is allowed to shake hands with more than one person at once.)

K11.3. Estonian Mathematical Olympiad 2002 Final Round 10th Grade 5/5.

The teacher writes numbers 1 at both ends of the blackboard. The first student inserts a 2 in the middle between them; each next student inserts the sum of each two adjacent numbers already on the blackboard between them (hence there are numbers 1, 3, 2, 3, 1 on the blackboard after the second student; 1, 4, 3, 5, 2, 5, 3, 4, 1 after the third student etc.) Find the sum of all numbers on the blackboard after the n-th student.

K11.4. Austrian Mathematical Olympiad (Final Round, Day 2): June 2002 2/3.

In a road network, the start of which we can see illustrated, the vertices in the middle row are labeled 1, 4, 7, … . The upper vertices are labeled 2, 5, 8, …, and the lower vertices are labeled 3, 6, 9, … . How many paths exist, leading from „1” to „3n + 1”, such that points are only passed through in increasing order?



K11.5. Bulgarian National Olympiad 1995, Third Round

Let $n > 1$ be a positive integer. Find the number of permutations $(a_1, a_2, \ldots, a_n)$ of the numbers $1, 2, \ldots, n$ with the following property: there exists exactly one index $i \in \{1, 2, \ldots, n - 1\}$ such that $a_i > a_{i+1}$.

Recursion in combinatorics 2.

K12.1. Konhauser Poblemfest April 2001 Canada

When Mark climbs a staircase, he ascends either 1, 2, or 3 stairsteps with each stride, but in no particular pattern from one foot to the next. In how many ways can Mark climb a staircase of 10 steps? (Note that he must finish on the top step. Two ways are considered the same if the number of steps for each stride are the same; that is, it does not matter whether he puts his best or his worst foot forward first.) Suppose that a spill has occured on the 6^{th} step and Mark wants to avoid it. In how many ways can he climb a staircase without stepping on the 6^{th} step?

K12.2. Japanese Mathematical Olympiad 2002 First Round

A disk divided into 7 congruent sectors is given. We have 4 colored pencils of red, blue, yellow, green, respectively. Suppose we want to paint each sector with one color out of the four. We may use the same color several times, and do not have to use all the colors. But it is required to paint the adjecent sectors with different colors.
How many different ways of painting are there?
Here, we identify two ways of paintings if one of those coincides with the other when the given circle is rotated around the center.

K12.3. Bulgarian Mathematical Competition February 2001

Let A_n be the number of sequences from 0's and 1's of length n, such that no four consequtive element equal 0101. Find the parity of A_{2001}.

K12.4. Estonian Mathematical Olympiad 2002 Final Round 11th Grade 5/5.

John built a robot that moves along the border of a regular octagon, passing each side of the octagon in exactly 1 minute. The robot begins its movement in some vertex A of the octagon, and further on reaching each vertex can either continue its way in the same direction, or turn around and continue in the opposite direction. In how many different ways can the robot move so that after n minutes it will be in the vertex B opposite to A?

K12.5. Irish Mathematical Olympiad 1999: First Day (Time: 3 hours) 5/5.

Three numbers $a < b < c$ are said to be in arithmetic progression if $c – b = b – a$.
Define a sequence u_n, $n = 0, 1, 2, \dots$ as follows: $u_0 = 0$, $u_1 = 1$ and for each $n \ge 1$, u_{n+1} is the smallest positive integer such that $u_{n+1} > u_n$ and $\{u_0, u_1, \dots ,u_n, u_{n+1}\}$ contains no three elements which are in arithmetic progression. Find u_{100}.

Number Theory in combinatorics 1.

K13.1. Russian Mathematical Olympiad 2002 IV-th (District) round 8-th form 1/8.

Does there exist a 9x2002 matrix with positive integer entries such that the sum of its entries in each row and the sum of its entries in each column are prime numbers?

K13.2. Mathematical national competition in Croatia 2002: 1st grade 4/4.

A „Wheel of Fortune” is divided in 30 parts and the numbers 50, 100, 150, … , 1500 (in some order) are written on these parts. Prove that there exist three consecutive parts, such that the sum of numbers written on them is at least 2350.

K13.3. Flanders Mathematics Olympiad 1998 Final Round 3/4.

Determine all 3x3 magic squares.

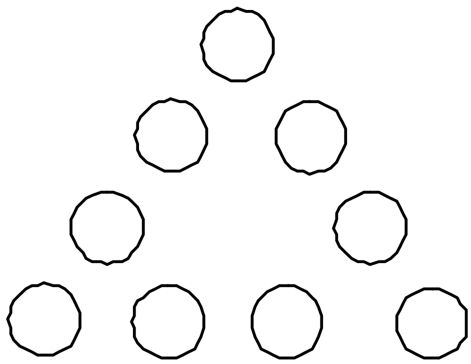
Definition: A nxn magic square is a nxn matrix formed with all the integers $1, 2, \dots , n^2$ and such that the sums of the elements in each row, in each column and in both diagonals are equal.

K13.4. Estonian Mathematical Olympiad 2002 Final Round 11th Grade 4/5.

Let a_1, a_2, a_3, a_4, a_5 be real numbers such that at least N of the sums $a_i + a_j$, where $i < j$, are integers. Find the greatest value of N for which it is possible that not all of the sums $a_i + a_j$ are integers.

K13.5. Asian Pacific Mathematical Olympiad March 2000 2/5.

Given the following triangular arrangement of circles, each of the numbers 1, 2, … , 9 is to be written into one of these circles, so that each circle contains exactly one of these numbers and
1) the sums of the four numbers on each side of the triangle are equal;
2) the sums of the squares of the four numbers on each side of the triangle are equal.
Find all ways in which this can be done.



Number Theory in combinatorics 2.

K14.1. Irish Mathematical Olympiad May 2002: Test 2 (Time: 3 hours) 1/5.

A 3xn grid is filled as follows: the first row consists of the numbers from 1 to n arranged from left to right in ascending order. The second row is a cyclic shift of the top row. Thus the order goes i, i + 1, … , n – 1, n, 1, 2, … , i – 1 for some i. The third row has the numbers 1 to n in some order, subject to the rule that in each of the n columns, the sum of the three numbers is the same.
For which values of n is it possible to fill the grid according to the above rules?
For an n for which this is possible, determine the number of different ways of filling the grid.

K14.2. Estonian Spring Open Contest: February 2002: Juniors 5/5.

For which positive integers n is it possible to write n real numbers, not all equal to 0, on a circle so that each of these numbers is equal to the absolute value of the difference of its two neighbours?

K14.3. Asian-Pacific Mathematics Olympiad April 2001 (4 hours) 2/5.

Find the largest positive integer N so that the number of integers in the set {1, 2, … , N} which are divisible by 3 is equal to the number of integer which are divisible by 5 or 7 (or both).

K14.4. Math Olympiad in Slovenia 2002 Final Round Grade Four 2/4.

Let S = {a_1, a_2, … , a_n} where a_i are different positive integers. The sum of all numbers of any proper subset of the set S is not divisible by n. Prove that the sum of all numbers of the set S is divisible by n.

K14.5. Hong Kong (China) Mathematical Olympiad 1999 3/4.

Let s, t be given nonzero integers, and let (x, y) be any ordered pair of integers. A move changes (x, y) to (x + t, y – s). The pair (x, y) is good if after some (may be zero) number of moves it describes a pair of integers that are not relatively prime.
a) Determine if (s, t) is a good pair.

b) Show that for any s and t there is pair (x, y) which is not good.

Number Theory in combinatorics 3.

K15.1. Lithuanian Mathematical Olympiad 1998

n is a positive integer of ten digits in the decimal system. The top digit of n is equal to the number of 0's. Similarly, the (k + 1)-th digit from then top is equal to the number of k's in n ($0 \le k \le 9$). Determine all such n's.

K15.2. Mandelbrot Competition December 1997 Canada

If a group of positive integers has a sum of n, what is the greatest product the group can have?

K15.3. Estonian Mathematical Olympiad 2002 Final Round 9th Grade 3/5.

Let $a_1, a_2, \dots, a_n$ be a pairwise distinct real numbers and m be the number of distinct sums $a_i + a_j$ (where $i \neq j$). Find the least possible value of m.

K15.4. Japanese Mathematical Olympiad 1992 First Round

Let $A = \{1, 2, \dots, 10\}$. How many maps $f: A \to A$ are there which satisfies the following:
a) $f^{30}(x) = x$ for every $x \in A$.
b) If $1 \le k \le 29$, there exists an $a \in A$ such that $f^k(a) \neq a$.

K15.5. German National Mathematics Competition 1998 Second Round

Let $M = \{1, 2, 3, \dots, 10000\}$. Prove that one can find 16 subsets of M with the following property: for each $z \in M$ there exist eight of these subsets having exactly z as common element.

Number Theory in combinatorics 4.

K16.1. Lithuanian Mathematical Olympiad 1998

A set of 1998 different numbers has such a property: if replace each number by the sum of other numbers, we get the same set of numbers. Prove that the product of all numbers is negative.

K16.2. Russian Mathematical Olympiad 2002 IV-th (District) round 9-th form 5/8.

The numbers 1, 2, … , 60 are arranged along a circle in an arbitrary order. Is it possible that the sum of any two numbers such that there is exactly one number between them is divisible by 2, the sum of any two numbers such that there are exactly two numbers between them is divisible by 3, and the sum of any two numbers such that there are exactly six numbers between them is divisible by 7?

K16.3. Estonian Mathematical Olympiad 2002 Final Round 10th Grade 3/5.

John takes seven positive integers $a_1, a_2, \dots, a_7$ and writes the numbers a_ia_j, $a_i + a_j$ and $|a_i - a_j|$ for all $i \neq j$ on the blackboard. Find the greatest possible number of distinct odd integers on the blackboard.

K16.4. Mathematical Olympiads' Correspondence Program 1996 Canada

Find all possible finite sequences $(w_0, w_1, \dots, w_n)$ of nonnegative integers with the property that, for each i = 0, 1, 2, … , n, the integer i appears w_i times and no other integer appears.

K16.5. Nordic Mathematics Contest 1998

For which positive integers n does there exist a permutation $(x_1, x_2, \dots, x_n)$ of numbers 1, 2, … , n such that the number $x_1 + x_2 + \dots + x_k$ is divisible by k for every $k \in \{1, 2, \dots, n\}$?

Cominatorial Geometry 1.

K17.1. South Africa, Stellenbosch Camp December 1998: Test 1 (Time: 3 hours) 5/5.

A regular 21-sided polygon is inscribed in a circle. Is it possible to choose five of its vertices in such a way as to define a pentagon, with sides and diagonals all of different lengths?

K17.2. Mathematical Olympiads' Correspondence Program 1996 Canada

Let n be a positiv integer exceeding 5. Given are n complanar points for which no two of the distances between pairs are equal. Suppose that each point is connected to the point nearest to it with a line segment. Prove that no point is connected to more than five others.

K17.3. Mathematical Competition 1997 Lithuania

A square with the diagonal d is divided into m rectangles having the diagonals $d_1, d_2, \dots, d_m$. Prove that $d_1^2 + d_2^2 + \dots + d_m^2 \geq d^2$.

K17.4. Albanian Mathematical Olympiad March 2002 12th class 4/5.

An infinite set S of coplanar points is given, such that every three of them are not collinear and any two of them are at least 1 cm apart. Does there exist a division of S in two disjoint infinite subsets R and B such that inside every triangle with vertices in R there is at least one point of B and inside every triangle with vertices in B there is at least one point of R?

K17.5. Asian-Pacific Mathematics Olympiad April 2001 (4 hours) 5/5.

Find the greatest integer n, such that there are $n + 4$ points A, B, C, D, $X_1, \dots, X_n$ in the plane with $AB \neq CD$ that satisfy the following condition: for each i = 1, 2, … , n triangles ABX_i and CDX_i are congruent.

Cominatorial Geometry 2.

K18.1. Estonian Mathematical Olympiad 2002 Final Round 10th Grade 4/5.

Find the maximum length of a broken line on the surface of a unit cube, such that its links are the cube's edges and diagonals of faces, the line does not intersect itself and passes no more than once through any vertex of the cube, and its endpoints are in two opposite vertices of the cube.

K18.2. South Africa, Potchefstroom Camp July 2001: Test 1 (Time: 4 hours 15 m.) 1/5.

Show that it is always possible to arrange 2001 points around a unit circle so that every two points are a rational distance apart.

K18.3. Turkish IMO Selection Ex. March 1999 (Second Day - Time: 4.5 hours) 3/3.

Prove that the plane is not a union of the inner regions of finitely many parabolas. (The outer region of a parabola is the union of the lines not intersecting the parabola. The inner region of a parabola is the set of points of the plane that do not belong to the outer region of the parabola.)

K18.4. Iranian Mathematical Olympiad 2002 First Round 3/6. Time: 2x4 hours

Find all natural numbers n for which there exist n unit squares in the plane with horizontal and vertical sides such that the obtained figure has at least 3 symmetry axes.

K18.5. Proposed IMO 1996

Suppose we have an axb rectangle and a cxd rectangle with $a < c \le d < b$ and $ab < cd$. Prove that the first rectangle can be placed within the second one if and only if

$$\left(b^2 - a^2\right)^2 \le (bd - ac)^2 + (bc - ad)^2.$$

Cominatorial Geometry 3.

K19.1. Mathematical competition in Lithuania October 2001

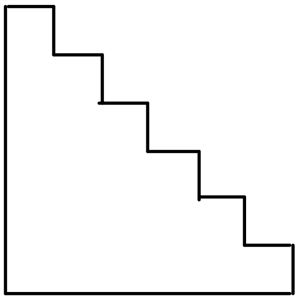
At least how many points in the plane with both integer coordinates can be coveredby a square of side 2.1?

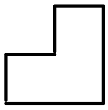
K19.2. Japanese Mathematical Olympiad 1991 First Round

How many planes are there which pass through at least three midpoints of segments of a cube?

K19.3. Estonian Mathematical Olympiad 1999 Final Round 11th Grade 4/5.

For which values of n it is posible to cover the side wall of staircase of n steps (n = 6 in the figure) with plates on the figure? The width and height of each step is 1, the dimensions of plate are 2x2 and from the corner there is cut out a 1x1 square piece.





K19.4. Mathematical Olympiads' Correspondence Program 1996 Canada

Given n points in the plane, not all on a line, such that the areas of the triangles defined by any three of them are less than 1, prove that the points can be covered by a triangle of area 4.

K19.5. Spanish Mathematical Olympiad 2002 modified problem

Se consideran 2002 segmentos en el plano tales que la suma de sus longitudes es la unidad. Probar que existe una recta r tal que la suma de las longitudes de las proyecciones de los 2002 segmentos dados sobre r es menor que $\frac{\sqrt{2}}{2}$.

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On the plane, 2002 segments are drawn such that the sum of their lengths is 1. Prove that there exists a line h such that the sum of the lengths of the projections on h is less than $\frac{\sqrt{2}}{2}$.

Colourings 1.

K20.1. Lithuanian Mathematical Olympiad 1998

A white square nxn is divided into n^2 unit squares. Any n of these squares are coloured black. We have to cut out a white rectangle of the area $S \geq n$. Is it always possible to do this if
a) $n = 7$; b) $n = 8$?

K20.2. Japanese Mathematical Olympiad 1991 Final Round

On a rectangle chess board of the size 10 x 14, the squares are colored white and black alternatively. We write 0 or 1 in every squares so that every rows and every columns contain odd pieces of 1. Prove that the total number of 1 in black squares is even.

K20.3. Russian Mathematical Olympiad 2002 IV-th (District) round 8-th form 2/8.

All cells in the 9x9 square are painted either in blue or red. Two cells are called neighbors if their intersection is a point. Show that there exists a cell having exactly two red neighbors or exactly two blue neighbors, or both.

K20.4. Spanish Mathematical Olympiad 2002 Final round 6/6.

En un polígono regular H de 6n + 1 lados (n entero positivo), r vértices se pintan de rojo y el ersto de azul. Demostrar que el número de triángulos isósceles que tienen sus tres vértices del mismo color no depende del mondo de distribuir los coloers en los vértices de H.

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The vertices of a regular 6n + 1 polygon H are coloured with red and blue. Prove that the number of isosceles triangles having their vertices identically coloured does not depend on the colouring.

K20.5. Romanian Selection Test 1997

The vertices of a regular dodecagon (12 sided polygon) are coloured either red or blue. Find the number of all possible colourings which do not contain monochromatic regular subpolygons.

Colourings 2.

K21.1. Hong Kong (China) Mathematical Olympiad 1999 2/4.

The base of a pyramid is a convex polygon with 9 sides. Each of the diagonals of the base and each of the edges on the lateral surface of the pyramid is coloured either black or white. Both colours are used. (Note that the sides of the base are not coloured.) Prove that there are three segments coloured the same colour which form a triangle.

K21.2 South Africa, Stellenbosch Camp December 2000: Test 2 (Time: 3.5 hours) 6/7.

Prove that it is not possible to label the points of an infinite lattice with labels A, B, C , D so that each unit square in the lattice has four different labels, and that each row or column contains each letter at least once.

K21.3 Japanese Mathematical Olympiad 1992 First Round

Let $A = \{(x, y) \in Z^2 \mid 1 \le x \le 20, 1 \le y \le 20\}$ and $B = \{(x, y) \in Z^2 \mid 2 \le x \le 19, 2 \le y \le 19\}$. Every points in A are colored by red or blue. The number of red points in A is 219. 180 red points in A are contained in B. Four points (1, 1), (1, 20), (20, 1), (20, 20) are colored by blue. We connect two colored lattice points of distance 1 by a colored segment as the following:
a) If both points are red, we connect them by a red segment.
b) If both points are blue, we connect them by a blue segment.
c) If one is red and the other is blue, we connect them by a black segment.
Now, there are 237 black segments of length 1. How many blue segments of length 1 are there?

K21.4. A similar problem from **1973 (Daniel Gold competition, grade 10, first round)**

A grid has 30 horizontal and 30 vertical lines. Out of the 900 grid points 627 are red, 273 are blue. Along the perimeter there are only 2 red points. The segments of the grid are coloured: between two red points the segment is red, between two blue the segment is blue. The remaining 101 segments are black. How many red segments are there?

K21.5. Serbian Mathematical Olympiad 1999 3rd form (Time: 4 hours) 4/5.

Given n vertical and n horizontal lines in the plane, intersecting at n^2 points. The lines are coloured blue, red or green. The intersection of two blue lines is blue, two red lines is red, two green lines is green, the intersection of blue and red lines is green, red and green lines is blue and green and blue lines is red. In that way, we obtain n^2 coloured points. Find the number of distinct colourings of these n^2 points that can be obtained this way.

Graphs

K22.1. South Africa, Rhodes Camp April 2001: Test 1 (Time: 4.5 hours) 3/3.

In a dance, a group S of 1994 students stand in a big circle. Each student claps the hands of each of his two neighbours a number of times. For each student x, let f(x) be the total number of times x claps the hands of his neighbours. As an example, suppose there are 3 students, A, B and C. A claps hands with B two times, B claps hands with C three times and C claps hands with A five times. Then f(A) = 7, f(B) = 5 and f(C) = 8.

a) Prove that $\{f(x): x \in S\} \neq \{n: n \text{ is an integer}, 2 \leq n \leq 1995\}$.

b) Find an example in which $\{f(x): x \in S\} = \{n: n \text{ is an integer}, n \neq 3, 2 \leq n \leq 1996\}$.

K22.2. Mathematical Olympiads' Correspondence Program 1996 Canada

There are nine people in a room. Two of any three know each other. Show that four people can be found in the room such that any two of them know each other.

K22.3. Irish Mathematical Olympiad May 2002: Test 1 (Time: 3 hours) 2/5.

a) A group of people attends a party. Each person has at most three acquaintances in the group, and if two people do not know each other, then they have a mutual acquaintance in the group. What is the maximum number of people present?

b) If, in addition, the group contains three mutual acquaintances (i.e., three people each of whom knows the other two), what is the maximum number of people?

K22.4. Balkan Mathematical Olympiad April 2002 1/4.

Let $A_1, A_2, \dots, A_n$ ($n \geq 4$) be points on the plane such that no three of them are collinear. Some pairs of distinct points among $A_1, A_2, \dots, A_n$ are connected by line segments in such a way that each point is connected to at least three others. Prove that there exist $k > 1$ and distinct points $X_1, X_2, \dots, X_{2k}$ in $\{A_1, A_2, \dots, A_n\}$ so that for each $1 \leq i \leq 2k - 1$, X_i is connected to X_{i+1} and X_{2k} is connected to X_1.

K22.5. Japanese Mathematical Olympiad 1998 Final Round

In a country, there are 1998 aiports. For any three aiports A, B, C, at least one of this pairs AB, BC, AC is not connected by any airline. (The airlines operate direct flights only.) How many airlines are there at most?