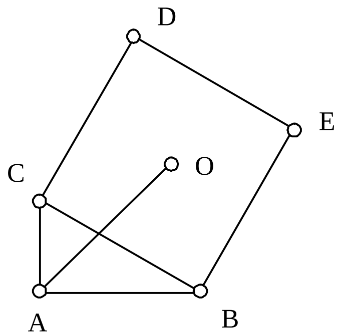
General Geometry 1.

G1.1. Auckland Mathematical Olympiad 1998: Division 1 3/5.

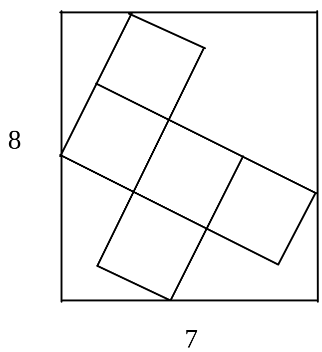
A square BCDE is constructed on the hypotenuse BC of a right-angled triangle ABC as shown in the figure.



Let O be the centre of the square. Given that the legs AB and AC have lengths m and n, respectively, find the distance AO.

G1.2. Estonian Autumn Open Contest: October 2001: Juniors 1/5.

A figure consisting of five equal-sized squares is placed as shown in a rectangle of size 7x8 units. Find the side length of the squares.



G1.3. Mathematical city competition in Croatia 2002: 2nd grade 1/4.

Let K be the midpoint of the hypotenuse AB of a right triangle ABC and let M be the point on the leg BC, such that BM = 2MC. Prove that $\angle MAB = \angle MKC$.

G1.4. Bulgarian Mathematical Competition March 2001

Given a square ABCD of side length 1. Point $M \in BC$ and point $N \in CD$ are such that the perimeter of the triangle MCN is 2. Find $\angle MAN$.

G1.5. Asian Pacific Mathematical Olympiad March 2000 3/5.

Let ABC be a triangle. Let M and N be the points in which the median and the angle bisector, respectively, at A meet the side BC. Let Q and P be the points in which the perpendicular at N to NA meets MA and BA, respectively, and O the point in which the perpendicular at P to BA meets AN. Prove that QO is perpendicular to BC.

General Geometry 2.

G2.1. Estonian Spring Open Contest: February 2002: Seniors 3/5.

Let ABCD be a rhombus with $\angle DAB = 60°$. Let K, L be points on its sides AD and DC and M a point on the diagonal AC such that KDLM is a paralelogram. Prove that triangle BKL is equilateral.

G2.2. Mathematical Olympiads' Correspondence Program 1996 Canada

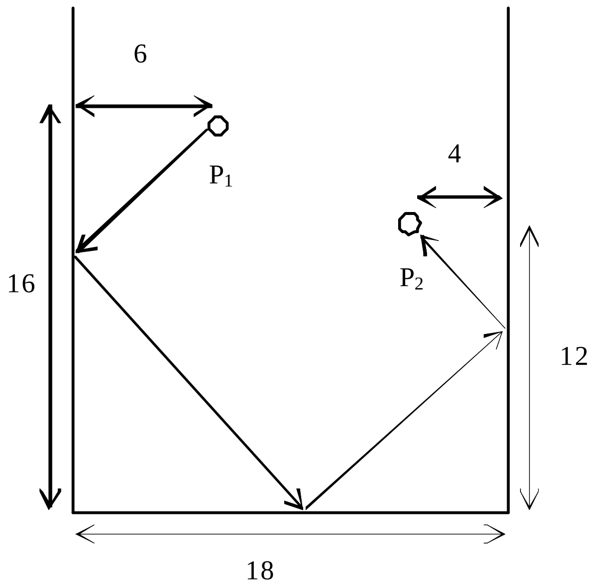
Let P be a point inside the triangle ABC such that $\angle PAC = 10°$, $\angle PCA = 20°$, $\angle PAB = 30°$ and $\angle ABC = 40°$. Determine $\angle BPC$.

G2.3. Math Olympiad in Slovenia 2002 Final Round Grade One 3/4.

Let M be the midpoint of the base AB of the trapezium ABCD. E is an interior point of the segment AC such that the lines BC and ME intersect at the point F, the lines FD and AB intersect at G and the lines DE and AB intersect at H. Prove that M is the midpoint of the segment GH.

G2.4. Flanders Mathematics Olympiad 1998 Final Round 4/4.

The figure represents three sides of a billiard table. A white ball is lying at position P_1 and the red ball is lying at position P_2. The dimensions given are expressed in dm. The white ball is played towards the red ball in such a way that it first touches all three sides of the table (see figure). Determine the minimal distance that this white ball covers.



G2.5. Mathematical county competition in Croatia 2002: 1st grade 3/4.

Inside a unit square, all isosceles triangles whose base is a side of the square, and whose vertex is the midpoint of the opposite side, are drawn. Find the area of the octagon determined by the intersection of these four triangles.

Triangle-geometry 1.

G3.1. Sharp Calculator Competition January 2001

ABCD is a rectangle and M is the midpoint of CD. A perpendicular BH is dropped onto AM from B. Prove that triangle BHC is isoceles.

G3.2. Spanish Mathematical Olympiad 2002 First round 2/8.

En el triángulo ABC, la bisectriz trazada desde A divide al lado opuesto en dos segmentos, de los que conocemos uno: BT = 572 m. Si dicha bisectriz corta a la medina BM en los segmentos BD = 200 m y DM = 350 m, calcula los lados a y c de dicho triángulo.

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In triangle ABC, the angle bisector from A meets the opposite side at point T and the median BM at point D. Let BT = 572, BD = 200 and DM = 350. Find the sides a and c.

G3.3. Macedonian Mathematical Competition 2002 II Round III Class 1/4.

Let ABC be a triangle such that $\angle A = \frac{\pi}{7}$, $\angle B = \frac{2\pi}{7}$, $\angle C = \frac{4\pi}{7}$, AB = c, BC = a and CA = b.

Prove that $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$.

G3.4. Bulgarian National Olympiad 1997

Given the triangle ABC, let M and N be the feet of the bisectors of $\angle ABC$ and $\angle ACB$ on AC and AB respectively. The ray MN intersects the circumcircle of triangle ABC at the point D. Prove that $\frac{1}{BD}=\frac{1}{AD}+\frac{1}{CD}$.

G3.5. Mathematical Olympiads' Correspondence Program 1996 Canada

Let M and O be the orthocentre and the circumcentre, respectively, of triangle ABC. Let N be the mirror reflection of M through O. Prove that the sum of the squares of the sides of the triangles NAB, NBC and NCA are equal.

Triangle-geometry 2.

G4.1. Olimpíada Brasileira de Matemática 2001 Segunda Fase Nível 3 2/6.

No triângulo ABC, a mediana e a altura relativas ao vértice A dividem o ângulo $\angle BAC$ em trẹs ângulos de mesma medida. Determine as medidas dos ângulos do triângulo ABC.

G4.2. Mathematical Competition 1997 Lithuania

On the sides AC and BC of the triangle ABC points N and M are marked such that $\frac{AN}{CN}=3$ and $\frac{BM}{CM}=2$. O is the point of intersection of lines AM and BN. Find the ratio $\frac{AO}{MO}$.

G4.3. Estonian IMO Slection Test 2002 First Day 2/3.

Consider an isosceles triangle KL_1L_2 with $|KL_1| = |KL_2|$, and let KA, L_1B_1, L_2B_2 be its angle bisectors. Prove that $\cos B_1AB_2 < \frac{3}{5}$.

G4.4. British Mathematical Olympiad 1997

In the acute-angled triangle ABC, CF is an altitude, with F on AB, and BM is a median, with M on CA. Given that BM = CF and $\angle MBC = \angle FCA$, prove that the triangle ABC is equilateral.

G4.5. Australian Mathematical Olympiad

The lines joining the three vertices of triangle ABC to a point in its plane cut the sides opposite vertices A, B, C in the points K, L, M respectively. A line through M parallel to KL cuts BC at V and AK at W. Prove that VM = MW.

Triangle-geometry 3.

G5.1. Spanish Mathematical Olympiad 2002 First round 6/8.

En el triángulo acutángulo ABC, AH, AD y AM son, respectivamente, la altura, la bisectriz y la mediana que parten desde A, estando H, D y M en el lado BC. Si las longitudes de AB, AC y MD son, respectivamente, 11, 8 y 1, calcula la longitud del segmento DH.

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In the acute triangle ABC, segments AH, AD and AM are, respectively, the altitude, the angle bisector and the median from A, and the lengths of AB, AC and MD are, respectively, 11, 8 and 1. Determine the length of segment DH.

G5.2. Macedonian Mathematical Competition 2002 I Round II Class

In the triangle ABC the incenter and the circumcenter are symmetrical with respect to the side AB. Find the angles of the triangle.

G5.3. Mathematical national competition in Croatia 2002: 2nd grade 3/4.

The lengths of the sides of two triangles are a, b, c and a’, b’, c’ and the opposite angles are α, β, γ and α', β', γ'. If the equalities $\alpha + \alpha' = \pi$ and $\beta = \beta'$ hold, prove that $aa' = bb' + cc'$.

G5.4. Estonian Spring Open Contest: February 2002: Juniors 3/5.

In a triangle ABC we have $|AB| = |AC|$ and $\angle BAC = \alpha$. Let $P \neq B$ be a point on AB and Q a point on the altitude drawn from A such that $|PQ| = |QC|$. Find $\angle QPC$.

G5.5. Mathematical Olympiads’ Correspondence Program 1996 Canada

The altitude from A of the triangle ABC intersects the side BC in D. A circle touches BC in D, intersects AB in M and N and intersects AC in P and Q. Prove that $\frac{AM+AN}{AC} = \frac{AP+AQ}{AB}$.

Triangle-geometry 4.

G6.1. British Mathematical Olympiad 1998 Round 1 5/5.

In triangle ABC, D is the midpoint of AB and E is the point of trisection of BC nearer to C. Given that $\angle ADC = \angle BAE$ find $\angle BAC$.

G6.2. Hellenic Math. Competition April 2002 (Final Selection Ex. for Juniors) 3/4.

Let ABC be a triangle with $\angle A = 60°$ and $AB \neq AC$. Let also AD be the bisector of the angle $\angle A$. The straight line e perpendicular to AD at A, intersects the extension of BC at the point E so that $BE = AB + AC$. Find the angles $\angle B$ and $\angle C$ of the triangle ABC.

G6.3. Mathematical national competition in Croatia 2002: 3rd grade 1/4.

A triangle with acute angles $\alpha = \angle BAC$ and $\beta = \angle CBA$ is given. Externally, isosceles triangles ACD and BCE with bases AC and BC, and vertex angles $\angle ADC = \beta$ and $\angle BEC = \alpha$, are drawn. Let O be the center of the circle circumscribed about the triangle ABC. Prove that DO + EO equals the perimeter of the triangle ABC if and only if $\angle ACB$ is right angle.

G6.4. South Africa, Rhodes Camp April 2001: Test 1 (Time: 4.5 hours) 2/3.

ABC is a triangle with $\angle A > 90°$. On the side BC, two distinct points P and Q are chosen such that $\angle BAP = \angle PAQ$ and $BP \cdot CQ = BC \cdot PQ$. Calculate the size of $\angle PAC$.

G6.5. The Monthly Problem Sets 1999

Let the triangle ABC have $\angle A = 30°$. Let K be its circumcentre and I its incentre. Choose a point D on BA and a point E on CA such that BD = CE = BC. Show that KI is perpendicular and equal to DE.

Triangle-geometry 5.

G7.1. Lithuanian Mathematical Olympiad 1998

A triangle with different sides is given. The angle bisector also halves the angle between the median and the height. Prove that the triangle is a rectangular one.

G7.2. Estonian Mathematical Contests 1996

Let a, b, c be the sides of a triangle and α, β, γ the opposite angles, respectively. Prove that if the inradius of the triangle is r, then $a \cdot \sin\alpha + b \cdot \sin\beta + c \cdot \sin\gamma \geq 9r$.

G7.3. Math Olympiad in Slovenia 2002 Final Round Grade Four 3/4.

Let A’ be the foot of the altitude on the side BC of the acute triangle ABC. The circle with diameter AA’ intersects the side AB at the points A and D, and the side AC at the points A and E. Prove that the circumcenter of triangle ABC lies on the line, determined by the altitude on the side DE of triangle ADE.

G7.4. German National Mathematics Competition 1999 First Round

Let AC be a segment in the plane and B an interior point of AC. Draw the counterclockwise oriented isosceles triangles ABS_1, BCS_2 and CAS_3, all having basis angles of 30°. Prove that $S_3S_2S_1$ is an equilateral triangle.

G7.5. Iranian Mathematical Olympiad 2002 First Round 2/6. Time: 2x4 hours

ABC is an acute triangle. Construct triangles A’BC, B’AC, C’AB outwardly on the sides of ABC such that $\angle A'BC = \angle B'AC = \angle C'BA = 30°$, $\angle A'CB = \angle B'CA = \angle C'AB = 60°$. Show that if N is the midpoint of BC, then B’N is perpendicular to A’C’.

Incidences 1.

G8.1. South Africa, Stellenbosch Camp December 2000: Test 4 (Time: 3.5 hours) 4/7.

In triangle ABC, cevians AD, BE and CF meet in O. If AO = 23, BO = 24, CO = 29, DO = 7 and EO = 8, determine FO.

G8.2. Math Olympiad in Slovenia 1998 Final Round Grade Three 3/4.

A rectangle ABCD with $|AB| > |AD|$ is given. The circle with center B and radius $|AB|$ intersects the line DC in the points E and F.
a) Prove that the circumcircle of the triangle EBF is tangent to the circle with diameter AD.
b) Denote the point of contact by G. Prove that the points D, G and B are collinear.

G8.3. Czech and Slovak Mathematical Olympiad December 2001: First Round

Let I be the center of the incircle of a given triangle ABC and P, Q the feet of the perpendiculars from the vertex C to the lines bisecting the interior angles BAC and ABC, respectively. Show that the lines AB and PQ are parallel.

G8.4. Austrian Mathematical Olympiad (Final Round, Day 1): June 2002 3/3.

Let ABCD and AEFG be similar cyclic quadrilaterals, whose vertices are labeled counter-clockwise. Let P be the second common point of the circumcircles of the quadrilaterals different from A. Show that P must lie on the line connecting B and E.

G8.5. Hong Kong (China) Mathematical Olympiad 1999 1/4.

PQRS is a cyclic quadrilateral with $\angle PSR = 90°$; H, K are the feet of the perpendiculars from Q to PR, PS (suitably extended if necessary), respectively. Show that HK bisects QS.

Incidences 2. - Three lines through a point

G9.1. Mathematical county competition in Croatia 2002: 3rd grade 1/4.

A square ABCD and a point P are given. Prove that the lines through the points B, C, D, A perpendicular to the lines AP, BP, CP, DP, respectively, are concurrent.

G9.2. South Africa, Stellenbosch Camp December 2000: Test 5 (Time: 4 hours) 1/3.

In the triangle ABC, the incircle touches BC at D, CA at E and AB at F. The midpoints of BC, CA and AB are A', B' and C' respectively. P, Q and R are chosen on EF, FD and DE respectively such that $A'P \perp EF$, $B'Q \perp FD$ and $C'R \perp DE$. Prove that A'P, B'Q and C'R are concurrent.

G9.3. Mathematical Competition 1990 New Zealand

In the acute angled triangle PQR, angle Q is 60°. Prove that the bisector of one of the angles formed by the altitudes drawn from P and R is passing through the centre of the circumcircle.

G9.4. Estonian Mathematical Contest 1998 Final round

Let A_1, B_1 and C_1 be the midpoints of the sides BC, CA and AB of the triangle ABC respectively and let A_2, B_2 and C_2 be the midpoints of the segments B_1C_1, C_1A_1 and A_1B_1

respectively. Let the incenters of the triangles B_1AC_1, C_1BA_1 and A_1CB_1 be respectively A_3, B_3 and C_3. Prove that the straight lines A_2A_3, B_2B_3 and C_2C_3 intersect in one point.

G9.5. Brazilian Mathematical Olympiad 2001 Second day 2/3.

Em um quadrilátero convexo, a altura em relação a um lado é definida como a perpendicular a esse lado passando pelo ponto médio do lado oposto. Prove que as quatro alturas tęm um ponto comum se e somente se o quadrilátero é inscritível, isto é, se e somente se existe uma circunferęncia que contém seus quatro vértices.

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In a convex quadrilateral, the altitude relative to a side is defined to be the line perpendicular to this side through the midpoint of the opposite side. Prove that the four altitudes have a common point if and only if the quadrilateral is cyclic, that is, if and only if, there exists a circle which contains its four vertices.

Incidences 3. – Three points on a line

G10.1. Estonian Open Contest 1999 (11^{th} and 12^{th} grade)

On the side BC of the triangle ABC a point D different from B and C is chosen so that the bisectors of the angles ACB and ADB intersect on the side AB. Let D' be the symmetrical point to D with respect to the line AB. Prove that the points C, A and D' lie on the same line.

G10.2. Czech and Slovak Mathematical Olympiad January 2002: Second Round

Given are a circle k and a quadrilateral ABCD inscribed into k. Its diagonal BD is not a diameter of the circle. Show that the intersection of the lines tangent to k at the points B and D lies on the line AC if and only if $AB\cdot CD = AD\cdot BC$.

G10.3. Balkan Mathematical Olympiad for Juniors 2002 2/4.

Two circles k_1 and k_2 of different radii intersect at A and B. Let B_1 and B_2 respectively the symmetric point of B with respect to the centers of k_1 and k_2 respectively and M the the midpoint of the segment B_1B_2. Take points M_1 on k_1 and M_2 on k_2 such that $\text{arc}(AB_1M_1) = \text{arc}(ABM_2) < 180°$. Show that $\angle MM_1B = \angle MM_2B$.

G10.4. Hellenic Math. Competition 2001 3/4. (Selection examination for the IMO 2002)

An acute angle triangle is given. Let M and N be interior points on the sides AC and BC, respectively, and K be the midpoint of the segment MN. The circumcircles of triangles CAN and BCM meet for a second time at the point D. Prove that the line CD passes through the circumcircle of the triangle ABC if and only if the perpendicular bisector of AB passes through K.

G10.5. Iranian Mathematical Olympiad 2002 Second Round 2/6. Time: 2x4 hours

Let AB be a diameter of a circle k. Suppose that l_a, l_b are tangent lines to k respectively at A, B. C is an arbitrary point on k. BC meets l_a at K, let the bisector of $\angle CAK$ meets CK at H. Let

M be the midpoint of the arcCAB and S be another intersection point of HM and k. Let T be the intersection of l_b and the tangent line to k at M. Show that S, T, K are collinear.

Incidences 4.

G11.1. South Africa, Potchefstroom Camp July 2001: Test 5 (Time: 4.5 hours) 1/4.

Let acute angled triangle ABC have altitudes AA', BB', CC', and orthocentre H. Let A_1, B_1, C_1 be points on AA', BB', CC' respectively (not their extensions), such that $|ABC_1| + |BCA_1| + |CAB_1| = |ABC|$. Prove that the circumcircle of $A_1B_1C_1$ is passing through H.

G11.2. The Second Selection Examination for the IMO, Romania 1996

A semicircle with center O and diameter AB is given. On the line AB a point M is chosen such that $|AM| > |BM|$ and that M does not lie on the segment AB. A line through M intersects the arcAB in points C and D such that $|CM| < |DM|$. The circumcircles of triangles AOD and OBC intersect each other in points O and K. Prove that the lines OK and KM are perpendicular to each other.

G11.3. British Mathematical Olympiad 1998 Round 2 2/4.

A triangle ABC has $\angle BAC > \angle BCA$. A line AP is drawn so that $\angle PAC > \angle BCA$ where P is inside the triangle. A point Q outside the triangle is constructed so that PQ is parallel to AB and BQ is parallel to AC. R is the point on BC (separated from Q by the line AP) such that $\angle PRQ = \angle BCA$. Prove that the circumcircle of ABC touches the circumcircle of PQR.

G11.4. Irish Mathematical Olympiad 1999: First Day (Time: 3 hours) 3/5.

Let D, E, F be points on the sides BC, CA, AB, respectively, of triangle ABC such that AD is perpendicular to BC, BE is the angle-bisector of $\angle B$ and F is the midpoint of AB. Prove that AD, BE, CF are concurrent if and only if $a^2(a - c) = (b^2 - c^2)(a + c)$, where a, b, c are the lengths of the sides BC, CA, AB, respectively, of triangle ABC.

G11.5. Iranian Mathematical Olympiad 2002 Third Round 3/6. Time: 2x4 hours

Let ABC be a triangle. The incircle of ABC touchs BC at A'. Let AA' meet the incircle at P. CP and BP meet the incircle of ABC at N, M respectively. Show that AA', BN, CM are concurrent.

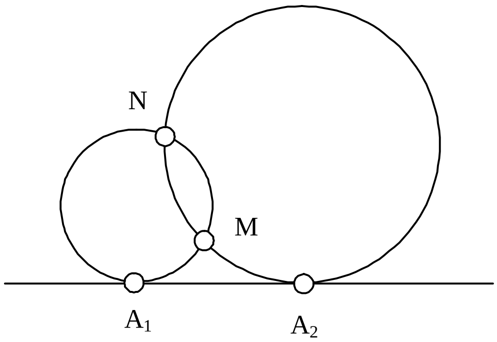
Geometry of the circle 1.

G12.1. Mathematical competition in Lithuania October 2001

It is known that parallel sides of the trapezium are 4 and 16 and the trapezium is such that it is possible to inscribe and circumscribe the circles. Find the radii of these circles.

G12.2. Auckland Mathematical Olympiad 1998: Division 2

Two circles C_1 and C_2 of radii r_1 and r_2 touch a line at points A_1 and A_2, as shown in the figure below.



The circles intersect at points M, N. Prove that the circumradius of the triangle A_1MA_2 does not depend on the length of A_1A_2 and is equal to $\sqrt{r_1r_2}$.

G12.3. Math Olympiad in Slovenia 1998 Final Round Grade Two 3/4.

The circle K with center O and an exterior point A are given. The line AO intersects the circle K in B and C, and the tangent line through A touches the circle in D. Choose an arbitrary point E on the line BD such that the point D lies between B and E. The circumcircle of the triangle DCE intersects the line AB in C and F, and the line AD in D and G. Prove that the lines BD and FG are parallel to each other.

G12.4. Pan-African Mathematics Olympiad July 2001: Day 2 (Time: 4.5 hours) 3/3.

S_1 is a semicircle with centre O and diameter AB. A circle C_1 with centre P is drawn tangent at O to AB and tangent to S_1. A semicircle S_2 is drawn with centre Q on AB, tangent to C_1 and to S_1. A circle C_2 with centre R is drawn, internally tangent to S_1 and externally tangent to S_2 and C_1. Prove that OPRQ is a rectangle.

G12.5. South Africa, Stellenbosch Camp December 2000: Test 3 (Time: 3.5 hours) 6/7.

ABCD is a convex quadrilateral. Show how to construct a point E such that the triangle ABE is similar to the triangle CDE.

Geometry of the circle 2.

G13.1. Italian Mathematical Olympiad May 2002 3/6.

Let A and B be two points of the plane and let M be the middle point. Let r be a line and R and S be the projections of A and B to r. Assuming that A, M and R are not collinear, prove that the circumcircle of triangle AMR has the same radius as the circumcircle of BSM.

G13.2. Balkan Mathematical Olympiad for Juniors 2002 1/4.

Let ABC be an isosceles triangle with AC = BC. Let also P be a point of the arc AB (not including C). If CD is perpendicular to PB, $D \in PB$, prove that PA + PB = 2PD.

G13.3. British Maths Olympiad 2000 Round 2

Two intersecting circles C_1 and C_2 have a common tangent which touches C_1 at P and C_2 at Q. The two circles intersect at M and N, when N is nearer to PQ than M is. The line PN meets the circle C_2 again at R. Prove that MQ bisects angle PMR.

G13.4. Estonian IMO Slection Test 2002 Second Day 1/3.

Let ABCD be a cyclic quadrilateral such that $\angle ACB = 2\angle CAD$ and $\angle ACD = 2\angle BAC$. Prove that $|CA| = |CB| + |CD|$.

G13.5. South Africa, Potchefstroom Camp July 1999: Test 4 (Time: 4.5 hours) 1/3. (Proposed to WFNMC)

In triangle ABC the incircle touches sides AB and AC at P and Q, respectively. The line through the incentre, parallel to BC, intersects AB and AC at K and L, respectively. Lines KQ and LP intersect at S. Segment SN is the altitude in triangle KLS. Show that $\angle PNS = \angle QNS$.

Geometry of the circle 3.

G14.1. Mathematical Competition 1992 New Zealand

Two circles have a common external tangent of length 26 and a common internal tangent of length 22. Find the product of the radii of the two circles. (The two internal tangents cross in the region between the circles.)

G14.2. British Maths Olympiad 2000 Round 2

Two intersecting circles C_1 and C_2 have a common tangent which touches C_1 at P and C_2 at Q. The two circles intersect at M and N, when N is nearer to PQ than M is. Prove that the triangles MNP and MNQ have equal areas.

G14.3. Mathematical competition in Lithuania October 2001

In the triangle ABC M is the midpoint of the side AB, O is the centre of circumscribed circle, $\angle COM = 90°$. Prove that $|\angle ABC - \angle BAC| = 90°$.

G14.4. South Africa, Potchefstroom Camp July 2001: Test 2 (Time: 4.5 hours) 2/4.

Let ABCD be a cyclic quadrilateral and R the radius of the circumcircle. Let a, b, c and d be the lengths of the sides of the quadrilateral ABCD and K its area. Prove that

$R^2 = \frac{(ab+cd)(ac+bd)(ad+bc)}{16K^2}$. Deduce that $R \geq \frac{(abcd)^{\frac{3}{4}}}{\sqrt{2}K}$.

G14.5. Iranian Mathematical Olympiad 2002 Second Round 5/6. Time: 2x4 hours

Let M, N be points on the side BC of a given triangle ABC such that BM = CN (M lies between B, N). P, Q are located respectively on AN, AM such that $\angle PMC = \angle MAB$, $\angle QNB = \angle NAC$. Prove that $\angle QBC = \angle PCB$.

Geometry of the circle 4.

G15.1. Mathematical competition in Lithuania October 2001

The chord corresponding to an arc of 60° divides the circle into two regions. Into the smaller part there is a square inscribed. Find the side of this square if radius of the circle is R.

G15.2. British Mathematical Olympiad December 2001: Round 1 (Time: 3.5 hours) 2/5.

The diagonals AC, BD of the cyclic quadrilateral ABCD meet at Q. The sides DA, extended beyond A and CB, extended beyond B, meet at P. Given that CD = CP = DQ, prove that $\angle CAD = 60°$.

G15.3. Mathematical Competition 1990 New Zealand

A point M is chosen inside the square ABCD in such a way that $\angle MAC = \angle MCD = x$. Find $\angle ABM$.

G15.4. Math Olympiad in Slovenia 1998 Final Round Grade Four 3/4.

Let D be the foot of the altitude on the hypotenuse BC of the right triangle ABC. The line through the centers of the incircles of triangles ABD and ADC intersects the legs of ABC in the points E and F. Prove that A is the center of the circumcircle of triangle DEF.

G15.5. South Africa, Rhodes Camp April 1999: Test 4 (Time: 4.5 hours) 2/3. (IMO 1985 Question 1)

A circle has its centre O on the side AB of a cyclic quadrilateral ABCD. The other three sides are tangent to the circle. Prove that AD + BC = AB.

Geometry of the circle 5.

G16.1. Mathematical Competition 1997 Lithuania

The parallel sides of a trapezium are 16 and 6, respectively. It is possible to inscribe a circle into it. Is it possible that the radius of this circle is 10?

G16.2. Asian-Pacific Mathematics Olympiad March 1999 (4 hours) 4/5.

Let k_1 and k_2 two circles intersecting at P and Q. The common tangent, closer to P, of k_1 and k_2 touches k_1 at A and k_2 at B. The tangent of k_1 at P meets k_2 at C, which is different from P, and the extension of AP meets BC at R. Prove that the circumcircle of triangle PQR is tangent to BP and BR.

G16.3. Estonian Mathematical Contests 1996

Let H be the orthocentre of an obtuse triangle ABC and A_1, B_1, C_1 arbitrary points taken on the sides BC, AC, AB, respectively. Prove that the tangents from the point H to the circles with diameters AA_1, BB_1, CC_1 are equal.

G16.4. Bulgarian Mathematical Competition March 2001

Given a convex quadrilateral ABCD such that $OA = \frac{OB \cdot OD}{OC + OD}$, where O is the intersection of its diagonals. The circumcircle of the triangle ABC intersects the line BD at point Q. Prove that CQ bisects $\angle DCA$.

G16.5. South Africa, Potchefstroom Camp July 1999: Test 3 (Time: 4.5 hours) 1/3. (Shortlisted for IMO 1998)

Let M and N be points inside triangle ABC such that $\angle MAB = \angle NAC$ and $\angle MBA = \angle NBC$. Prove that $\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1$.

Geometry of the circle 6.

G17.1. Russian Mathematical Olympiad 2002 IV-th (District) round 9-th form 3/8.

Let O be the circumcenter of an isosceles triangle ABC (AB = BC). A point M lies on the segment BO, the point M' is symmetric to M with respect to the midpoint of AB. The point K is the intersection of M'O and AB. The point L on the side BC is such that $\angle CLO = \angle BLM$. Show that the points O, K, B, L are on the same circle.

G17.2. German National Mathematics Competition 1998 Second Round

Given a triangle ABC and a point P on its side AB with the following properties:
a) $|BC| = |AC| + \frac{1}{2}|AB|$,
b) $|AP| = 3|PB|$.
Prove that $\angle PAC = 2\angle CPA$.

G17.3. Estonian Mathematical Olympiad 2002 Final Round 12th Grade 4/5.

All vertices of a convex quadrilateral ABCD lie on a circle ω. The rays AD, BC intersect in a point K and the rays AB, DC intersect in point L. Prove that the circumcircle of triangle AKL is tangent to ω if and only if the circumcircle of triangle CKL is tangent to ω.

G17.4. Iranian Mathematical Olympiad 1995

Points D and E are situated on the sides AB and AC of triangle ABC in such a way that $DE \parallel BC$. Let P be an arbitrary point inside the triangle ABC. Lines PB and PC intersect DE at F and G, respectively. Let O_1 be the circumcentre of triangle PDG and let O_2 be that of PFE. Show that $AP \perp O_1O_2$.

G17.5. Balkan Mathematical Olympiad April 2002 3/4.

Two circles with different radii intersect at two points A and B. The common tangents of these circles are MN and ST where the points M, S are on one of the circles and N, T are on the other one. Prove that the orthocenters of the triangles AMN, AST, BMN and BST form a rectangle.

Locus

G18.1. Estonian Mathematical Olympiad 1999 Final Round 10th Grade 5/5.

Let C be an interior point of line segment AB. Equilateral triangles ADC and CEB are constructed to the same side from AB. Find the locus of the midpoint of the segment DE.

G18.2. Mathematical Olympiads' Correspondence Program 1996 Canada

Let A, B, C be three distinct points of the plane for which AB = AC. Describe the locus of the points P for which $\angle APB = \angle APC$.

G18.3. Japanese Mathematical Olympiad 1998 First Round

ABCD is a trapezoid with AB = BC = DA = 1, CD = $1 + \sqrt{2}$ and the lines AB and CD are parallel. Let E be a point on AD such that if we fold this trapezoid along a line passing through E, A is moved onto a point on CD. Determine the maximum length of DE.

G18.4. Bulgarian Mathematical Competition March 2001

Given a square ABCD of side length 1. Point $M \in BC$ and point $N \in CD$ are such that the perimeter of the triangle MCN is 2. If P is the foot of the perpendicular from A to MN, find the locus of the point P.

G18.5. Vietnamese Mathematical Olympiad March 2002: First Day 2/3.

In the plane is given an isosceles triangle ABC (AB = AC). A variable circle (O) with center O on the line BC, passes through A but does not touch the lines AB, AC. Let M, N be respectively the second points of intersection of the circle (O) with the lines AB, AC. Find the locus of the orthocenter of triangle AMN.

Area 1.

G19.1. Japanese Mathematical Olympiad 1992 First Round

ABC is a regular triangle, and n > 6. Let D, E, F be the points which divide the segment BC, CA, AB respectively in the ratio 3 : (n – 3). Assume that the area of the triangle determined by three lines AD, BE, CF is equal to $\frac{4}{49}|ABC|$. Determine n.

G19.2. Serbian Mathematical Olympiad 1999 1st form (Time: 4 hours) 1/5.

A convex hexagon ABCDEF is given. Both of the diagonals AD and BE are halving the area of the hexagon. Prove that the quadrilateral BDEA is a trapezoid.

G19.3. Estonian Mathematical Olympiad 1999 Final Round 10th Grade 3/5.

The incircle of the triangle ABC, with the center I, touches the sides AB, AC and BC in the points K, L and M respectively. Points P and Q are taken on the sides AC and BC respectively, such that $|AP| = |CL|$ and $|BQ| = |CM|$. Prove that the difference of areas of APIQB and CPIQ is equal to the area of CLIM.

G19.4. South Africa, Talent Search 2000: Senior round 5, 5/5.

In triangle ABC, right-angled at B, a semicircle is drawn with its diameter on hypotenuse AC so that it is tangent to AB and BC. The radius of this semicircle is r_b. Two further semicircles are drawn so that their diameters lie on sides AB and BC. Each has one end of its diameter at vertex B and each is tangent to hypotenuse AC. The radii of these semicircles are r_c and r_a. If the incircle has radius r, prove that $\frac{2}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}$.

G19.5. Macedonian Mathematical Competition 2002 I Round I Class

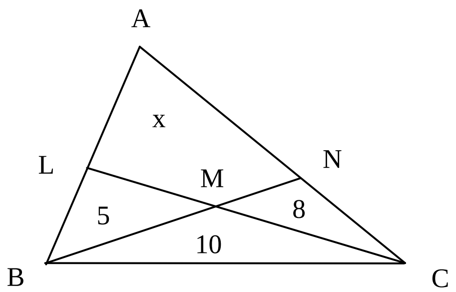
Let ABC be a given triangle and k be the inscribed circle in it. Three tangents on k paralell to the sides of ABC are drawn. Each of the tangents cuts a triangle from the triangle ABC. Let r be the radius of k and r_1, r_2, r_3 be the radii of the inscribed circles in each of the three triangles cut from the triangle ABC. Prove that $r_1 + r_2 + r_3 = r$.

Area 2.

G20.1. Japanese Mathematical Olympiad 1998 First Round

ABCD is the rectangle on the xy-plane with the vertices A(3, 0), B(3, 2), C(0, 2), and D(0, 0). Let S be the set of the points (u, v) which satisfy $0 \le ux + vy \le 1$ for every (x, y) in the rectangle ABCD. Determine the area of S.

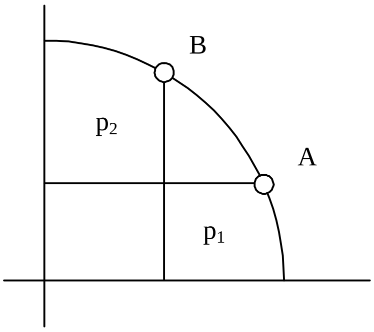
G20.2. Mathematical Competition 1991 New Zealand



In the triangle above, the areas of three smaller triangles are given. Find x, the area of quadrilateral ALMN.

G20.3. Math Olympiad in Slovenia 2002 Final Round Grade Three 3/4.

On the unit circle with center in the origin an arc with endpoints A and B in the first quadrant is chosen. Let p_1 be the area of the figure between the arc and its projection onto x-axis and let p_2 be the area of the figure between the arc and its projection onto y-axis (see figure). Prove that the sum $p_1 + p_2$ depends only on the length of the arc and not on its position.

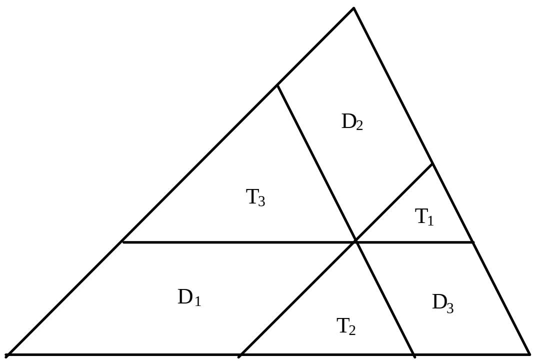


G20.4. Serbian Mathematical Olympiad 1999 2nd form (Time: 4 hours) 1/5.

Let AA_1, BB_1 and CC_1 be the altitudes of acute-angled triangle ABC and let A_2, B_2, C_2 be the points where the lines AA_1, BB_1, CC_1 intersect the circumscribed circle of the triangle ABC. Prove that $\frac{AA_2}{AA_1}+\frac{BB_2}{BB_1}+\frac{CC_2}{CC_1}=4$.

G20.5. South Africa, Potchefstroom Camp July 2001: Test 5 (Time: 4.5 hours) 4/4.

A triangle is divided into six regions labeled as in the figure, by lines parallel to the sides and concurrent at a point inside the triangle. Prove that $\frac{|T_1|}{|D_1|}+\frac{|T_2|}{|D_2|}+\frac{|T_3|}{|D_3|}\geq\frac{3}{2}$.



Geometric inequalities 1.

G21.1. Auckland Mathematical Olympiad 1997: Division 2 4/5.

Let r be the inradius and R be the circumradius of a right-angled triangle ABC. Let S be the area of ABC. Prove that r + R > $\sqrt{2S}$.

G21.2. Albanian Mathematical Olympiad March 2002 9th class 3/5.

The right triangle with area S is circumscribed the circle circumscribed the right triangle with area s. Prove that $\frac{S}{s} \geq 3 + 2\sqrt{2}$. When does equality hold?

G21.3. Czech and Slovak Mathematical Olympiad October 2001
(Problems for the take-home part)

If S is the area of a triangle with sides a, b, c, and T is the area of the triangle with sides a + b, b + c, c + a, then T ≥ 4S. Prove this and find when equality holds.

G21.4. South Africa, Potchefstroom Camp July 2001: Test 3 (Time: 4.5 hours) 1/3.

Let P be a point inside triangle ABC. Let r_1, r_2 and r_3 be the distances from P to the sides BC, CA and AB respectively. If R is the circumradius of triangle ABC, show that
$\left(\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3}\right)^2 \leq \frac{a^2 + b^2 + c^2}{2R}$.

G21.5. Canadian Mathematical Olympiad 1992. 3/1. Time: 3 hours

Suppose ABCD is a square and U and V are points on the interior of the sides AB and CD respectively. Let AV and DU intersect at P, and BV and CU intersect at Q. Determine all possible ways of selecting U and V so as to maximise the area of PUQV.

Geometric inequalities 2.

G22.1. South Africa, Stellenbosch Camp December 2000: Test 2 (Time: 3.5 hours) 3/7.

In triangle ABC, $a^3 + b^3 = c^3$. Prove that ∠C is acute.

G22.2. British Maths Olympiad 2000 Round 2

Triangle ABC has a right angle at A. Among all points P on the perimeter of the triangle, find the position of P such that AP + BP + CP is minimised.

G22.3. Hellenic Mathematical Olympiad February 2002 3/4.

A triangle ABC is given with ∠C > 10° and ∠B = ∠C + 10°. We consider points E, D on the segments AB, AC, respectively, such that ∠ACE = 10° and ∠DBA = 15°. Let Z ≠ A is a point of intersection of the circumcircles of the triangles ABD and AEC.
Prove that ∠ZBA > ∠ZCA.

G22.4. Proposed IMO 1997

Let ABCDEF be a convex hexagon such that AB = BC, CD = DE, EF = FA.

Prove that $\frac{BC}{BE}+\frac{DE}{DA}+\frac{FA}{FC}\geq\frac{3}{2}$. When does equality hold?

G22.5. Albanian Mathematical Olympiad March 2002 12th class 5/5.

Prove the following relations in any triangle ABC with inradius and exradius r and R, bisectors l_a, l_b, l_c of its angles A, B, C, and altitudes h_a, h_b, h_c from its vertices A, B and C, respectively:

a) $\frac{h_a}{l_a^2}+\frac{h_b}{l_b^2}+\frac{h_c}{l_c^2}=\frac{R+2r}{2Rr}$,

b) $\frac{1}{l_a}+\frac{1}{l_b}+\frac{1}{l_c}\leq\frac{1}{r}\sqrt{\frac{1}{2}+\frac{r}{R}}$.

Geometric inequalities 3.

G23.1. Ukrainan Mathematical Olympiad April 1998 10th grade

Let M be an internal point of the side AC of a triangle ABC, O is the intersection point of perpendiculars from midpoints of AM and MC to lines BC and AB respectively. Find the location of M such that the length of segment OM is minimal.

G23.2. Spanish Mathematical Olympiad 1999 National Round Second Day 2/3.

El baricentre del triángulo ABC es G. Denotamos por g_a, g_b, g_c las distancias desde G a los lados a, b y c respectivamente. Sea r el radio de la circunferencia inscrita. Probar que:

a) $g_a\geq\frac{2r}{3}, g_b\geq\frac{2r}{3}, g_c\geq\frac{2r}{3}$;

b) $\frac{g_a+g_b+g_c}{r}\geq 3$.

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The centroid of triangle ABC is G. Denote by g_a, g_b, g_c the distances from G to sides a, b and c respectively. Let r be the radius of the incircle. Prove that:

a) $g_a\geq\frac{2r}{3}, g_b\geq\frac{2r}{3}, g_c\geq\frac{2r}{3}$;

b) $\frac{g_a+g_b+g_c}{r}\geq 3$.

G23.3. Serbian Mathematical Olympiad 1999 2nd form (Time: 4 hours) 5/5.

A mathematician get lost in a forest of the shape of an infinitely long strip. If the width of the strip is 1 km, prove that he can get out of the woods by walking no more than $2\sqrt{2}$ km.

G23.4. Austrian Mathematical Olympiad April 2002 Qualifying Round 3/4.

The circumference of a convex hexagon ABCDEF is s and the circumferences of the triangles ACE and BDF are u and v, respectively.

a) Prove that $\frac{1}{2} < \frac{s}{u+v} < 1$ holds.

b) Determine whether 1 can be replaced by a smaller number, or $\frac{1}{2}$ by a larger number, such that the inequalities still hold for all convex hexagons.

G23.5. Estonian Mathematical Olympiad 2002 Final Round 12th Grade 3/5.

Prove that for positive real numbers a, b and c the inequality $2(a^4 + b^4 + c^4) < (a^2 + b^2 + c^2)^2$ holds if and only if there exists a triangle with side lengths a, b and c.

Vectors

G24.1. South Africa, Rhodes Camp April 2001: Test 3 (Time: 4.5 hours) 1/3.

Let ABC be an equilateral triangle, and P be a point on the incircle. Show that $PA^2 + PB^2 + PC^2$ is constant.

G24.2. Spanish Mathematical Olympiad 1999 Second Local Round First Day 3a/3.

Dado un triángulo ABC, con baricentro G. Prueba que para cualquier punto M del plano se verifica: $MA^2 + MB^2 + MC^2 \ge GA^2 + GB^2 + GC^2$, obteniéndose la igualdad si y solamente si M = G.

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Given is a triangle ABC with centroid G. Prove that for any point M of then plane $MA^2 + MB^2 + MC^2 \ge GA^2 + GB^2 + GC^2$, and equality holds if and only if M = G.

G24.3. Albanian Mathematical Olympiad March 2002 9th class 4/5.

The point H is the intersection of the altitudes of the triangle ABC. The points A_1, B_1, C_1 are chosen respectively on the rays HA, HB, HC such that $HA_1 = BC$, $HB_1 = CA$, $HC_1 = AB$. Prove that

a) $\overrightarrow{HA_1} + \overrightarrow{HB_1} + \overrightarrow{HC_1} = \vec{0}$.

b) The point H is the intersection of the medians of the triangle $A_1B_1C_1$.

G24.4. Estonian Mathematical Olympiad 1999 Final Round 11th Grade 3/5.

For the given triangle ABC, prove that a point X on the side AB satisfies the condition

$$\overrightarrow{XA} \cdot \overrightarrow{XB} + \overrightarrow{XC} \cdot \overrightarrow{XC} = \overrightarrow{CA} \cdot \overrightarrow{CB}$$

iff X is the foot of the altitude or of the median of the triangle ABC ($\vec{v} \cdot \vec{u}$ denotes the scalar product of vectors $\vec{v}$ and $\vec{u}$).

G24.5. Mathematical Olympiads' Correspondence Program 1996 Canada

The convex quadrilateral ABCD of area 2t has no parallel sides. Locate the point P_1 on CD such that P_1 is on the same side of AB as C and the area of the triangle ABP_1 is t. If P_2, P_3 and P_4 are similarly defined for the sides DA, AB and BC, respectively, prove that P_1, P_2, P_3 and P_4 are collinear.

Trigonometry 1.

G25.1. Mathematical city competition in Croatia 2002: 3rd grade 4/4.

Let H be the orthocenter of the acute triangle ABC. Prove that $BC\cdot \text{ctg}\angle CAB = AH$.

G25.2. Math Olympiad in Slovenia 1998 First Round Grade Three 3/4.

Choose the point D on the side BC of triangle ABC such that $|BD| = |AC| = 1$ and $\angle BAD = \frac{1}{3}\angle DAC = 30°$. Calculate the length of the segment CD.

G25.3. British Mathematical Olympiad February 2002: Round 2 (Time: 3.5 hours) 1/4.

The altitude from one of the vertices of an acute-angled triangle ABC meets the opposite side at D. From D perpendiculars DE and DF are drawn to the other two sides. Prove that the length of EF is the same whichever vertex is chosen.

G25.4. Thai Mathematical Olympiad 2001

Let C_1 be a circumcircle of radius r of a triangle ABC. Tangents to C_1 at A, B, C meet at P, Q, R. Assume that the perimeter of triangle PQR has length m.
Compute $\tan\angle ABC + \tan\angle BCA + \tan\angle CAB$ in terms of r and m.

G25.5. South Africa, Potchefstroom Camp July 2001: Test 5 (Time: 4.5 hours) 3/4.

Prove that, in triangle ABC, $\frac{\cos A}{1+\sin A}+\frac{\cos B}{1+\sin B}+\frac{\cos C}{1+\sin C}\ge 6-3\sqrt{3}$.

Trigonometry 2.

G26.1. Mathematical city competition in Croatia 2002: 4th grade 2/4.

On the legs of the acute angle α with the vertex A, two points D and E are given such that AD = m and AE = n. Through the points D and E lines perpendicular to the respective legs of he given angle are constructed. If the intersection F of these lines lies inside the given angle, prove that $\frac{DF}{EF}=\frac{n-m\cos\alpha}{m-n\cos\alpha}$.

G26.2. Estonian Autumn Open Contest: October 2001: Seniors 4/5.

In a triangle ABC we have $\angle B = 2\cdot\angle C$ and the angle bisector drawn from A intersects BC in a point D such that $|AB| = |CD|$. Find $\angle A$.

G26.3. Iranian Mathematical Olympiad 2002 First Round 5/6. Time: 2x4 hours

In a triangle ABC (AB > AC) the bisectors of $\angle B$, $\angle C$ meet the opposite sides respectively at P, Q. Also let I be the intersection point of these bisectors. If IP = IQ, determine the angle $\angle A$.

G26.4. Math Olympiad in Slovenia 1998 First Round Grade Four 3/4.

Let H be the orthocenter and T the centroid of an acute triangle ABC in which the segment HT is parallel to the side AB. Prove that $\mathrm{tg}\alpha\cdot\mathrm{tg}\beta = 3$ if α and β are the interior angles at A and B respectively.

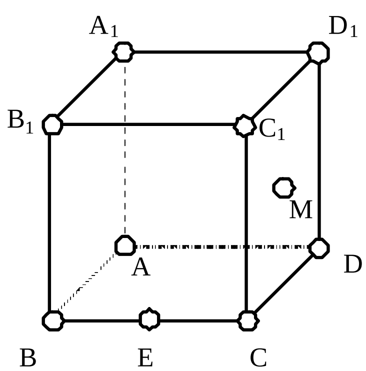
G26.5. Albanian Mathematical Olympiad March 2002 11th class 3/5.

A triangle ABC with sides a, b, c and respective angles α, β, γ is given. Prove the equivalence of the following propositions:
a) $\alpha = 2\gamma$,
b) $b = 4c\cos\left(30° + \frac{\alpha}{2}\right)\cos\left(30° - \frac{\alpha}{2}\right)$.

Solid geometry

G27.1. Flanders Mathematics Olympiad 1998 Final Round 2/4.

Given a cube with edges length 1; given the midpoint E of the edge BC and the midpoint M of the face CDD_1C_1 (see figure). Determine the area of the intersection of the cube with the plane determined by A, E and M.



G27.2. Ukrainan Mathematical Olympiad April 1998 11th grade

Two spheres with distinct radii are tangent externally at point P. Line segments AB and CD are given such that first sphere touches them at points A and C, the second sphere tangents them at points B and D. Let M and N be the orthogonal projections of the midpoints of segments AC and BD on the line of the centers of the given spheres. Prove that PM = PN.

G27.3. Macedonian Mathematical Competition 2002 I Round III Class

A plane intersects the lateral edges of a regular foursided prism in points A, B, C and D whose distance from the vertex of the prism is a, b, c and d, respectively. Prove that:
$\frac{1}{a}+\frac{1}{c}=\frac{1}{b}+\frac{1}{d}$.

G27.4. Serbian Mathematical Olympiad 1999 3rd form (Time: 4 hours) 1/5.

Given a tetrahedron ABCD. The edge AC is perpendicular to BC and AD is perpendicular to BD. Prove that cosine of the angle between the lines AC and BD is less than $\frac{CD}{AB}$.

G27.5. Mathematical Olympiads' Correspondence Program 1996 Canada

Let ABCD be a tetrahedron for which the sides AB, BC and CA have length a, while the sides AD, BD and CD have length b. Suppose that M and N are the midpoints of the sides AB and CD respectively. A plane passing through M and N intersects segments AD and BC in points P and Q.
a) Prove that AP : AD = BQ : BC.
b) Find the ratio of AP to AD if quadrilateral [MQNP] is minimum.