Elementary Algebra 1.

A1.1. Macedonian Mathematical Competition 2002 I Round I Class

Positive real numbers x and y satisfied the equality $x^2 + y^2 = 6xy$. Calculate the expression $\frac{x+y}{x-y}$.

A1.2. Mathematical county competition in Croatia 2002: 2nd grade 3/4.

Determine all real numbers a, such that all the solutions of the equation

$$\frac{a^2}{x(x+1)}+\frac{a^2}{(x+1)(x+2)}+\frac{a^2}{(x+2)(x+3)}+\frac{a^2}{(x+3)(x+4)}+\frac{a^2}{(x+4)(x+5)}=1$$

are real.

A1.3. Hellenic Mathematical Olympiad February 2002 (Juniors) 2/4.

In the Mathematical Competition of HMS (Hellenic Mathematical Society) take part boys and girls who are divided in to two groups: „Juniors” (at most is years old) and „seniors”.
The number of the boys taking part of this year competition is the 55% of the number of all participants. The ratio of the number of „junior boys” to the number of „senior boys” is equal to the ratio of the number of „juniors” to the number of „seniors”.
Find the ratio of the number of „junior boys” to the number of „junior girls”.

A1.4. Mathematical Competition 1997 Lithuania

Find all the pairs of real numbers a and b such that $ae^x + b = e^{ax + b}$ holds for all x.

A1.5. Pan-African Mathematics Olympiad July 2001: Day 2 (Time: 4.5 hours) 1/3.

Let $n \geq 1$ be an integer and $a > 0$ a real number. Find the number of solutions $(x_1, x_2, \dots, x_n)$ of the equation $\sum_{i=1}^{n}\left(x_i^2 + (a - x_i)^2\right) = na^2$ such that x_i belong to the interval [0, a], for i = 1, 2, … , n.

Elementary Algebra 2.

A2.1. Austrian Mathematical Olympiad April 2001: Regional Competition 2/3.

Determine all real solutions of the equation

$(x + 1)^{2001} + (x + 1)^{2000}(x - 2) + (x + 1)^{1999}(x - 2)^2 + \dots + (x + 1)^2(x - 2)^{1999} + (x + 1)(x - 2)^{2000} + (x - 2)^{2001} = 0.$

A2.2. Macedonian Mathematical Competition 2002 II Round I Class 3/4.

Let x, y, z be real numbers such that $\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}=1$.

Prove that $\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}=0$.

A2.3. Mathematical city competition in Croatia 2002: 2nd grade 2/4.

Let a and b real numbers, not equal to zero. Determine all solutions of the equation

$\frac{1}{a}+\frac{1}{b}+\frac{1}{x}=\frac{1}{a+b+x}$.

A2.4. Spanish Mathematical Olympiad 1999 Second Local Round Second Day 3/3.

Sean a, b y c números reales no nulos (con suma no nula) tales que: $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$.

Prueba que también se verifica: $\frac{1}{a^{1999}}+\frac{1}{b^{1999}}+\frac{1}{c^{1999}}=\frac{1}{a^{1999}+b^{1999}+c^{1999}}$.

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Let a, b and c be nonzero real numbers (with nonzero sum) such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$.

Prove that $\frac{1}{a^{1999}}+\frac{1}{b^{1999}}+\frac{1}{c^{1999}}=\frac{1}{a^{1999}+b^{1999}+c^{1999}}$.

A2.5. Albanian Mathematical Olympiad March 2002 11th class 1/5.

Calculate the sum

$$f\left(\frac{1}{2002}\right)+f\left(\frac{2}{2002}\right)+...+f\left(\frac{2001}{2002}\right)+f\left(\frac{2002}{2002}\right)+\ 2f\left(\frac{2002}{2001}\right)+2f\left(\frac{2002}{2000}\right)+...+2f\left(\frac{2002}{1}\right)$$

for the function $f(x)=\frac{2^{\frac{1+x}{x}}-2^x}{3}, \quad x\in R^*$.

Integer Part, Fractional Part 1.

([x] and {x} denote the integer and fractional part of x)

A3.1. Mathematical competition in Lithuania October 2001

Solve the equation $x^3 = 4 + [x]$.

A3.2. Estonian Autumn Open Contest: October 2001: Juniors 3/5.

Find all triples (x, y, z) of real numbers satisfying the system of equations:

$x + [y] + \{z\} = 200,2;$
$\{x\} + y + [z] = 200,1;$
$[x] + \{y\} + z = 200,0.$

A3.3. Albanian Mathematical Olympiad March 2002 10th class 2/5.

a) Prove that when $n = m^2 + m$, $m \in N$, then $\lfloor\sqrt{n}\rfloor = m$.
b) Find all the natural numbers n such that $\lfloor\sqrt{n}\rfloor$ divides n.

A3.4. Belarusian Selection Test 1997

Evaluate the sum $\left\lfloor\frac{1}{1997}\right\rfloor+\left\lfloor\frac{2}{1997}\right\rfloor+\left\lfloor\frac{2^2}{1997}\right\rfloor+...+\left\lfloor\frac{2^{1995}}{1997}\right\rfloor$.

A3.5. (Arany Dániel-verseny a speciális matematika osztályok számára, haladók, második (döntő) forduló, 1979. május 3.)

Határozzuk meg a következő összeg értékét (ahol {x} az x törtrészét jelöli):

$$\left\{\frac{1979\cdot 1+16}{111}\right\}+\left\{\frac{1979\cdot 2+16}{111}\right\}+...+\left\{\frac{1979\cdot 110+16}{111}\right\}+\left\{\frac{1979\cdot 111+16}{111}\right\}.$$

Integer Part, Fractional Part 2.

([x] and {x} denote the integer and fractional part of x)

A4.1. Austrian Mathematical Olympiad Beginners Competition: June 2002

Prove that there is no positive rational number x such that $x^{[x]} = \frac{9}{2}$ holds.

A4.2. Mathematical Competition 1997 Lithuania

Solve the equation $\sqrt{\left[-7x^2+3x+4\right]} = [2 - \sin x]$.

A4.3. Pan-African Mathematics Olympiad July 2001: Day 2 (Time: 4.5 hours) 2/3.

Calculate $\lfloor\sqrt{1}\rfloor+\lfloor\sqrt{2}\rfloor+...+\lfloor\sqrt{2001}\rfloor$.

A4.4. Mathematical Olympiads' Correspondence Program 1996 Canada

Prove that, for any non-negative integer n, the equation
$\left[n^{\frac{1}{2}}+(n+1)^{\frac{1}{2}}+(n+2)^{\frac{1}{2}}\right]=\left[(9n+8)^{\frac{1}{2}}\right]$ holds.

A4.5. Irish Mathematical Olympiad May 2002: Test 2 (Time: 3 hours) 4/5.

Let $\alpha = 2 + \sqrt{3}$. Prove that $\alpha^n - [\alpha^n] = 1 - \alpha^{-n}$, for n = 0, 1, 2, …

Equations of High Order 1.

A5.1. Mathematical national competition in Croatia 2002: 2nd grade 1/4.

Determine all the solutions of the equation

$(x^2 + 3x - 4)^3 + (2x^2 - 5x + 3)^3 = (3x^2 - 2x - 1)^3$.

A5.2. First Round of the Selection Test for IMO 1990 Japan

Determine the product of all the real roots of the equation $x^2 + 25x + 52 = 3\sqrt{x^2 + 25x + 80}$.

A5.3. Spanish Mathematical Olympiad 2002 First round 1/8.

Si p es un número real y las raíces de $x^3 + 2px^2 - px + 10 = 0$ están em progresión aritmética, halla dichas raíces.

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The roots of $x^3 + 2px^2 - px + 10 = 0$ are in arithmetical progression. Find them (p is a real number).

A5.4. Italian Mathematical Olympiad May 2002 4/6.

Find all values of n for which all solutions of the equation $x^3 - 3x + n = 0$ are integers.

A5.5. British Mathematical Olympiad 1998

a) Find a solution of the simultaneous equations

$xy + yz + xz = 12,$
$xyz = 2 + x + y + z$

in which all of x, y, z are positive, and prove that it is the only such solution.

b) Show that a solution exists in which x, y, z are real (possibly non-positive) and distinct.

Equations of High Order 2.

A6.1. Vietnamese Mathematical Olympiad March 2002: First Day 1/3.

Solve the equation $\sqrt{4 - 3\sqrt{10 - 3x}} = x - 2$.

A6.2. Bulgarian Mathematical Competition February 2001

Find all values of the real parameter a such that the equation $\log_x(x^2 + x + a)^2 = 4$ has unique solution.

A6.3. Mathematical county competition in Croatia 2002: 3rd grade 4/4.

Determine the angles α and β of the right triangle, if the following condition

$\mathrm{tg}\alpha + \mathrm{tg}\beta + \mathrm{tg}^2\alpha + \mathrm{tg}^2\beta + \mathrm{tg}^3\alpha + \mathrm{tg}^3\beta = 70$,

is satisfied. (It suffices to determine tgα and tgβ.)

A6.4. Mathematical Competition 1997 Lithuania

Find those values of *a* such that the equation $x^8 + ax^4 + 1 = 0$ have four roots in arithmetical progression.

A6.5. Japanese Mathematical Olympiad 1992 First Round

Let E be the curve defined by $y^2 = x^3 + 2691x - 8019$ in the xy-plane. Let P be the third intersection point of E and the line passing through (3, 9) and (4, 53). Determine the x-coordinate of P.

System of Algebraic Equations 1.

A7.1. Austrian Mathematical Olympiad April 2002 Qualifying Round 2/4.

Solve the following system of equations in real numbers:

$$2x_1 = x_5^2 - 23,$$
$$4x_2 = x_1^2 + 7,$$
$$6x_3 = x_2^2 + 14,$$
$$8x_4 = x_3^2 + 23,$$
$$10x_5 = x_4^2 + 34.$$

A7.2. Hellenic Mathematical Competition 2001 2/4. (Selection ex. for the IMO 2002)

Let *x*, *y*, *a* be a real numbers such that $x + y = x^3 + y^3 = x^5 + y^5 = a$.
Determine all the possible values of *a*.

A7.3. Czech and Slovak Mathematical Olympiad December 2001: First Round

Determine for which values of the real parameter p does the system of equations

$x^2 + 1 = (p + 1)x + py - z$,

$y^2 + 1 = (p + 1)y + pz - x,$
$z^2 + 1 = (p + 1)z + px - y,$

with unknowns x, y, z, have exactly one solution in the domain of the real numbers.

A7.4. Mathematical city competition in Croatia 2002: 2nd grade 3/4.

Prove that if $ax^3 = by^3 = cz^3$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 1$, then $\sqrt[3]{ax^2 + by^2 + cz^2} = \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}$.

A7.5. Iranian Mathematical Olympiad 1995

Let a, b, c be positive real numbers. Find all real numbers x, y, z such that
$x + y + z = a + b + c,$
$4xyz - (a^2x + b^2y + c^2z) = abc.$

System of Algebraic Equations 2.

A8.1. Irish Mathematical Olympiad 1999: Second Day (Time: 3 hours) 1/5.

Solve the system of (simultaneous) equations
$y^2 = (x + 8)(x^2 + 2),$
$y^2 - (8 + 4x)y + (16 + 16x - 5x^2) = 0.$

A8.2. Mathematical city competition in Croatia 2002: 4th grade 3/4.

Determine the real solutions of the system of equations

$x_1 + x_2 + \ldots + x_{2002} = 2002,$
$x_1^4 + x_2^4 + \ldots + x_{2002}^4 = x_1^3 + x_2^3 + \ldots + x_{2002}^3.$

A8.3. Hellenic Math. Competition April 2002 (Final Selection Ex. for Juniors) 1/4.

Let x, y, a be a real numbers such that $x + y = x^2 + y^2 = x^3 + y^3 = a$.
Determine all the possible values of a.

A8.4. Albanian Mathematical Olympiad March 2002 9th class 1/5.

The real numbers a and b satisfy the condition:

$a^3 - 3ab^2 = \sqrt{402}$,
$b^3 - 3a^2b = 40.$

Find $a^2 + b^2$.

A8.5. Belarusian Math. Competition, Minsk, 1995, 11th class

Find the real solutions of the following system of equations:

$\sqrt{x^2+y^2}+\sqrt{(x-4)^2+(y-3)^2}=5$,
$3x^2 + 4xy = 24$.

Inequalities 1.

A9.1. Mathematical competition in Lithuania October 2001

Find the minimum value of the expression $5(a^2 + b^2 + 2c^2) - 2(2ab + 6ac + bc - 2a + 3c)$, if a, b, c are real numbers.

A9.2. Estonian Spring Open Contest: February 2002: Seniors 2/5.

Let a, b be any real numbers such that $|a| \neq |b|$. Prove that $\left|\frac{a+b}{a-b}\right|^{ab} \geq 1$.

A9.3. Albanian Mathematical Olympiad March 2002 10th class 1/5.

The sum of the positive numbers x, y, z, t is 1.
a) Prove that $\frac{x^2}{x+y}+\frac{y^2}{y+z}+\frac{z^2}{z+t}+\frac{t^2}{t+x} \geq \frac{1}{2}$.
b) When does equality hold?

A9.4. Brazilian Mathematical Olympiad 2001 First day 1/3.

Prove que $(a + b)(a + c) \geq 2\sqrt{abc(a+b+c)}$ para quaisquer números reais positivos a, b e c.

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Prove that $(a + b)(a + c) \geq 2\sqrt{abc(a+b+c)}$ for all positive real numbers a, b and c.

A9.5. Australian Mathematical Olympiad 1986, Question 6

The numbers $a, b, a_2, a_3, \dots, a_{n-2}$ are all real, and $ab \neq 0$. All the roots of the equation

$$ax^n - ax^{n-1} + a_2x^{n-2} + \dots + a_{n-2}x^2 - n^2bx + b = 0$$

are real and positive. Prove that all the roots are equal.

Inequalities 2.

A10.1. Mathematical county competition in Croatia 2002: 1st grade 4/4.

Let *a* be a real number such that $a^5 - a^3 + a = 2$. Prove the inequalites: $3 < a^6 < 4$.

A10.2. Hellenic Mathematical Olympiad February 2002 1/4.

For the real numbers a, b, c with $bc \neq 0$ is given that $\frac{1-c^2}{bc} \geq 0$.
Prove that $10(a^2 + b^2 + c^2 - bc^3) \geq 2ab + 5ac$.

A10.3. Mathematical Competition 1997 Lithuania

Positive numbers a, b, and c satisfy the condition $a^2 + b^2 + c^2 = \frac{7}{4}$. Prove that
$\frac{1}{a} + \frac{1}{b} - \frac{1}{c} < \frac{1}{abc}$.

A10.4. Mathematical Competition 1990 New Zealand

The positive numbers a, b, c satisfy $a \geq b \geq c$ and $a + b + c \leq 1$. Prove that $a^2 + 3b^2 + 5c^2 \leq 1$.

A10.5. Asian Pacific Mathematical Olympiad March 2000 4/5.

Let n, k be given positive integers with $n > k$. Prove that

$$\frac{1}{n+1} \cdot \frac{n^n}{k^k (n-k)^{n-k}} < \frac{n!}{k!(n-k)!} < \frac{n^n}{k^k (n-k)^{n-k}}.$$

Inequalities 3.

A11.1. Irish Mathematical Olympiad 1999: First Day (Time: 3 hours) 1/5.

Find all real values of x which satisfy $\frac{x^2}{\left(x+1-\sqrt{x+1}\right)^2} < \frac{x^2+3x+18}{(x+1)^2}$.

A11.2. Manitoba Mathematical Contest, February 2001 Canada

If x, y and z are positive real numbers, prove that $(x + y - z)(x - y)^2 + z(x - z)(y - z) \geq 0$.

A11.3. Ukrainan Mathematical Olympiad April 1998 11th grade

For real numbers $x, y, z \in (0; 1]$ prove the inequality
$\frac{x}{1+y+zx} + \frac{y}{1+z+xy} + \frac{z}{1+x+yz} \leq \frac{3}{x+y+z}$.

A11.4. Mathematical competition in Lithuania October 2001

Function f(x) is defined for all real x and it admits real values. Assume
$f\left(\frac{x_1+x_2}{2}\right) \leq \frac{f(x_1)+f(x_2)}{2}$.

Does it follow that for all real values $f\left(\frac{x_1+x_2+x_3}{3}\right)\le\frac{f(x_1)+f(x_2)+f(x_3)}{3}$ holds?

A11.5. Hellenic Math. Competition 2001 (Selection Examination for the IMO 2002) 4/4.

Prove that the following inequality holds for every triplet (a, b, c) of non-negative real numbers with $a^2 + b^2 + c^2 = 1$:
$\frac{a}{b^2+1}+\frac{b}{c^2+1}+\frac{c}{a^2+1}\ge\frac{3}{4}\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2$. When does equality hold?

Inequalities 4.

A12.1. Mathematical Competition 1997 Lithuania

Integers x and y satisfy 4x + 5y = 7. Find the minimum of $5|x|-3|y|$.

A12.2. Irish Mathematical Olympiad 1998: First Day (Time: 3 hours) 1/5.

Show that if x a nonzero real number, then $x^8-x^5-\frac{1}{x}+\frac{1}{x^4}\ge 0$.

A12.3. Macedonian Mathematical Competition 2002 II Round II Class 1/4.

Let $p(t) = at^2 + bt + c$ be a polynomial with non negative coefficients.
Prove that $(p(xy))^2 \le p(x^2)p(y^2)$.

A12.4. Balkan Mathematical Olympiad for Juniors 2002 4/4.

Prove that $\frac{1}{b(a+b)}+\frac{1}{c(b+c)}+\frac{1}{a(a+c)}\ge\frac{27}{2(a+b+c)^2}$, for all positive numbers a, b, c.

A12.5. Vietnamese Mathematical Olympiad March 2002: Second Day 1/3.

Let a, b, c be real numbers such that the polynomial $P(x) = x^3 + ax^2 + bx + c$ has three real roots (not necessarily distinct). Prove that $12ab+27c\le 6a^3+10\left(a^2-2b\right)^{\frac{3}{2}}$.
When does equality hold?

Inequalities 5.

A13.1. South Africa, Potchefstroom Camp July 2001: Test 2 (Time: 4.5 hours) 1/4.

Let $x \ge 0$, $y \ge 0$ be real numbers with $x + y = 2$. Prove that $x^2y^2(x^2 + y^2) \le 2$.

A13.2. Lithuanian Mathematical Olympiad 1998

Four different real numbers a, b, c and d satisfy the conditions $\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}=4$, ac = bd. Find the maximum of $\frac{a}{c}+\frac{b}{d}+\frac{c}{a}+\frac{d}{b}$.

A13.3. Second Round of the Selection Test for IMO 1990 Japan

Let n > 2 be integer. Determine the maximum K and the minimum G such that for any positive real number $a_1, a_2, \dots, a_n$, the following inequality holds:

$$K < \frac{a_1}{a_1+a_2}+\frac{a_2}{a_2+a_3}+\dots+\frac{a_n}{a_n+a_1} < G.$$

A13.4. Irish Mathematical Olympiad 1999: Second Day (Time: 3 hours) 3/5.

Let a, b, c and d be positive numbers whose sum is 1. Prove that $\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}\ge\frac{1}{2}$, with equality if and only if $a = b = c = d = \frac{1}{4}$.

A13.5. Asian Pacific Mathematical Olympiad 1996

If a, b, c are the sides of a triangle, then prove that

$$\sqrt{a+b-c}+\sqrt{b+c-a}+\sqrt{c+a-b}\le\sqrt{a}+\sqrt{b}+\sqrt{c}.$$

Power Mean 1.

A14.1. Hellenic Mathematical Olympiad February 2002 (Juniors) 4/4.

Prove that $1\cdot 2\cdot 3\cdot\ldots\cdot 2002<\left(\frac{2003}{2}\right)^{2002}$.

A14.2. British Maths Olympiad 2000 Round 2

Given that x, y, z are positive real numbers satisfying xyz = 32, find the minimum value of $x^2 + 4xy + 4y^2 + 2z^2$.

A14.3. Mathematical national competition in Croatia 2002: 1st grade 2/4.

Prove that for every positive numbers a, b, c and for every nonnegative number p the inequality
$a^{p+2} + b^{p+2} + c^{p+2} \ge a^p bc + b^p ca + c^p ab$ holds.

A14.4. Albanian Mathematical Olympiad March 2002 12th class 3/5.

a) Prove the following inequality if a, b, c are positive numbers:

$$(a+b+c)+\sqrt{a^2+b^2+c^2}\le\frac{\sqrt{3}+1}{3\sqrt{3}}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(a^2+b^2+c^2\right).$$

b) When does equality hold?

A14.5. Estonian Mathematical Contest 1998 Selectional round

Let $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ be real numbers satisfying the conditions $x_1 \ge x_2 \ge \dots \ge x_n > 0$ and $y_1 \ge x_1$, $y_1y_2 \ge x_1x_2$, …, $y_1y_2\dots y_n \ge x_1x_2\dots x_n$. Prove that $y_1 + y_2 + \dots + y_n \ge x_1 + x_2 + \dots + x_n$.

Power Mean 2.

A15.1. Albanian Mathematical Olympiad March 2002 9th class 5/5.

Prove that for every three non-negative real numbers a, b, c such that a + b + c = 1, the following inequality holds: $7(ab + bc + ac) \le 2 + 9(abc)$. When does equality hold?

A15.2. Mathematical national competition in Croatia 2002: 2nd grade 2/4.

Let a, b, c be real numbers greater than 1. Prove the inequality

$$\log_a\left(\frac{b^2}{ac} - b + ac\right)\cdot\log_b\left(\frac{c^2}{ab} - c + ab\right)\cdot\log_c\left(\frac{a^2}{bc} - a + bc\right) \ge 1.$$

A15.3. Greek Mathematical Olympiad 2000 3/4.

Find the maximum value of k such that $\frac{xy}{\sqrt{(x^2+y^2)(3x^2+y^2)}} \le \frac{1}{k}$ for all positive numbers x and y.

A15.4. Austrian Mathematical Olympiad May 2001: National Competition (Day 1) 2/3.

Determine all triples of positive real numbers x, y and z such that both

$x + y + z = 6$ and

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2 - \frac{4}{xyz}$$

hold.

A15.5. Irish Mathematical Olympiad May 2002: Test 1 (Time: 3 hours) 4/5.

Let $0 < a, b, c < 1$. Prove that $\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \ge \frac{3\sqrt[3]{abc}}{1-\sqrt[3]{abc}}$.
Determine the case of equality.

Power Mean 3.

A16.1. Japanese Mathematical Olympiad 1998 First Round

Let x, y, and z be positive numbers. Determine the maximum value of $\frac{x^3y^2z}{x^6+y^6+z^6}$.

A16.2. Irish Mathematical Olympiad 1998: Second Day (Time: 3 hours) 2/5.

Prove that if a, b, c are positive real numbers, then

a) $\frac{9}{a+b+c} \le 2\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{a+c}\right)$;

b) $\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{a+c} \le \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$.

A16.3. Austrian Mathematical Olympiad Qualifying Round: April 2002 4/4.

Let $a_0, a_1, a_2, \ldots, a_{2002}$ be real numbers.

a) Prove that the smallest of the numbers $a_k(1 - a_{2002-k})$, $0 \le k \le 2002$, is not greater than $\frac{1}{4}$.

b) Does this also hold for the smallest of the numbers $a_k(1 - a_{2002-k})$, $1 \le k \le 2002$?

c) If $a_0, a_1, a_2, \ldots, a_{2002}$ are all positive real numbers, prove that the smallest of the numbers $a_k(1 - a_{2002-k})$, $0 \le k \le 2002$, is not greater than $\frac{1}{4}$.

A16.4. Albanian Mathematical Olympiad March 2002 11th class 5/5.

Prove that if a, b are positive numbers such that $a^2 + b^2 = 1$ and $a < b$, than

a) $\left(a^2\right)^{\frac{a}{b}} < 1 - ab$;

b) $a^a < b^b$.

A16.5. Mathematical Olympiads' Correspondence Program 1996 Canada

Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$\frac{x_1^2}{x_1+x_2}+\frac{x_2^2}{x_2+x_3}+\ldots+\frac{x_{n-1}^2}{x_{n-11}+x_n}+\frac{x_n^2}{x_n+x_1} \ge \frac{1}{2}(x_1 + x_2 + \ldots + x_n).$$

Power Mean 4.

A17.1. Austrian Mathematical Olympiad (Final Round, qualifying day): May 2002 2/4.

Determine the largest real number C, such that

$$\frac{((x+y)^2-6)((x-y)^2+8)}{(x-y)^2} \ge C$$

holds for all real numbers x and y ($x \neq y$) with $xy = 2$. For which ordered pairs (x, y) does equality occur?

A17.2. Selection Test for IMO 1990 Japan

If x, y, z are positive real numbers which satisfy $x + y + z = 1$, determine the minimum value of $\frac{1}{x}+\frac{4}{y}+\frac{9}{z}$.

A17.3. Canadian Open 1996

The sum of sixteen positive numbers is 100 and the sum of their squares is 1000. Prove that none of the sixteen numbers is greater than 25.

A17.4. Ukrainan Mathematical Olympiad April 1998 9th grade

Prove the inequality $\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ac}{1+c} \geq 3$ for positive real numbers a, b, c with $abc = 1$.

A17.5. Turkish Math. Olympiad December 1998 (First Day - Time: 4.5 hours) 2/3.

Prove that $(a + 3b)(b + 4c)(c + 2a) \geq 60abc$ for all real numbers $0 \leq a \leq b \leq c$.

Arithmetic Sequence

A18.1. Spanish Mathematical Olympiad 1999 First Local Round First Day 1/3.

Determina los posibles valores del primer término de una progresión aritmética para que cumpla, simultáneamente, las siguientes propiedades:
1) Todos los términos son positivos.
2) La diferencia está comprendida entre 0 y 1.
3) La suma de los 1999 primeros términos es exactamente 2000.

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Find the possible values of the first term of an arithmetic progression satisfying the following properties:
1) All terms are positive.
2) The difference is between 0 and 1.
3) The sum of the 1999 first terms is 2000.

A18.2. Russian Mathematical Olympiad 2002 IV-th (District) round 10-th form 1/8.

What is the maximum length of an arithmetic progression of positive integers $a_1, a_2, \dots, a_n$ with difference 2 and such that the number $a_k^2 + 1$ is prime for all $k = 1, 2, \dots, n$?

A18.3. Macedonian Mathematical Competition 2002 I Round IV Class

Let $a_1, a_2, \dots, a_n$ be an arythmetic progression. Prove that

$$\frac{1}{a_1 a_n}+\frac{1}{a_2 a_{n-1}}+\dots+\frac{1}{a_n a_1}=\frac{2}{a_1+a_n}\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}\right).$$

A18.4. Estonian Autumn Open Contest: October 2001: Seniors 2/5.

The side lengths of a triangle and the diameter of its incircle, taken in this order, form an arithmetic progression. Prove that the triangle is right-angled.

A18.5. Mathematical Competition 1991 New Zealand

Show that there are infinitely many terms of the arithmetic sequence 1, 14, 27, 40, … which are of the form 222…2.

Sequences

A19.1. Auckland Mathematical Olympiad 1998: Division 1 2/5.

The first number in a sequence is 7. The next one we obtain as follows: we compute the square of the previous number $7^2 = 49$, then compute the sum of the digits of this square and increase it by 1, i. e., the second number is 4 + 9 + 1 = 14. We repeat this procedure to get $14^2 = 196$ and the third number of the sequence, which is 1 + 9 + 6 + 1 = 17, and so on. What would be the 1999th number in this sequence?

A19.2. Olimpíada Brasileira de Matemática 2001 Segunda Fase Nível 3 6/6.

Seja $f(x) = \frac{x^2}{1+x^2}$. Calcule

$$\begin{aligned}
&f\left(\frac{1}{1}\right)+f\left(\frac{2}{1}\right)+f\left(\frac{3}{1}\right)+\dots+f\left(\frac{n}{1}\right)+\\
&+f\left(\frac{1}{2}\right)+f\left(\frac{2}{2}\right)+f\left(\frac{3}{2}\right)+\dots+f\left(\frac{n}{2}\right)+\\
&+f\left(\frac{1}{3}\right)+f\left(\frac{2}{3}\right)+f\left(\frac{3}{3}\right)+\dots+f\left(\frac{n}{3}\right)+\\
&+\dots+\\
&+f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+f\left(\frac{3}{n}\right)+\dots+f\left(\frac{n}{n}\right),
\end{aligned}$$

sendo n inteiro positivo.

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For a given positive integer n determine the value of the following sum:

$$f\left(\frac{1}{1}\right)+f\left(\frac{2}{1}\right)+f\left(\frac{3}{1}\right)+\ldots+f\left(\frac{n}{1}\right)+$$
$$+f\left(\frac{1}{2}\right)+f\left(\frac{2}{2}\right)+f\left(\frac{3}{2}\right)+\ldots+f\left(\frac{n}{2}\right)+$$
$$+f\left(\frac{1}{3}\right)+f\left(\frac{2}{3}\right)+f\left(\frac{3}{3}\right)+\ldots+f\left(\frac{n}{3}\right)+$$
$$+\ldots+$$
$$+f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+f\left(\frac{3}{n}\right)+\ldots+f\left(\frac{n}{n}\right).$$

A19.3. Japanese Mathematical Olympiad 1998 First Round

There are sufficiently many black stones and white stones. We pick up 10 stones, and align them under the rule that two black stones do not adjoin. How many such aligments are there?

A19.4. Mathematical national competition in Croatia 2002: 4th grade 4/4.

Let (a_n), $n \in N$ be an increasing sequence of postive integers. The term a_k of this sequence is called good if it can be written as the sum of some other (not necessarily different) terms of this sequence. Prove that all terms of this sequence, except for finite number of them, are good.

A19.5. Asian Pacific Mathematical Olympiad March 2000 1/5.

Compute the sum $S=\sum_{i=0}^{101}\frac{x_i^3}{1-3x_i+3x_i^2}$ for $x_i=\frac{i}{101}$.

Recursions 1.

A20.1. Estonian Spring Open Contest: February 2002: Juniors 4/5.

Define $a_1, a_2, \ldots a_n, \ldots$ as follows: $a_1 = 0$, $a_2 = 1$, $a_n = 5a_{n-1} - a_{n-2}$, for $n > 2$. For which n is a_n divisible by: a) 5; b) 15?

A20.2. British Maths Olympiad 2000 Round 2

For each positive integer k, define the sequence (a_n) by $a_0 = 1$, $a_n = kn + (-1)^n a_{n-1}$ for each $n \geq 1$. Determine all values of k for which 2000 is a term of the sequence.

A20.3. Mathematical Olympiads' Correspondence Program 1996 Canada

Let the sequence $\{u_n\}$ be defined recursively by $u_0 = 0$, $u_1 = 1$, $u_n = 1995u_{n-1} - u_{n-2}$ $(n \geq 2)$. Find all values of n exceeding 1 for which u_n is prime.

A20.4. Irish Mathematical Olympiad May 2002: Test 1 (Time: 3 hours) 4/5.

Let the sequence $a_1, a_2, a_3, \ldots$ be defined by $a_1 = 1$, $a_2 = 1$, $a_3 = 1$ and $a_{n+1}a_{n-2} - a_n a_{n-1} = 2$, for all $n \geq 3$. Prove that a_n is a positive integer for all $n \geq 1$.

A20.5. Balkan Mathematical Olympiad April 2002 2/4.

The sequence $a_1, a_2, \ldots, a_n, \ldots$ is defined by $a_1 = 20$, $a_2 = 30$, $a_{n+2} = 3a_{n+1} - a_n$, $n > 1$.
Find all positive integers n for which $1 + 5a_n a_{n+1}$ is a perfect square.

Recursions 2.

A21.1. Irish Mathematical Olympiad 1998: Second Day (Time: 3 hours) 4/5.

A sequence of real numbers x_n is defined recursively as follows: x_0, x_1 are arbitrary positive real numbers, and $x_{n+2} = \frac{1 + x_{n+1}}{x_n}$, n = 0, 1, 2, … Find x_{1998}.

A21.2. First Round of the Selection Test for IMO 1990 Japan

A sequence of numbers $\{a_n\}$ is defined by $a_1 = 1$ and $a_{n+1} = a_n + \frac{1}{a_n}$ for $n \geq 1$. Determine $[a_{100}]$.

A21.3. British Mathematical Olympiad 1998 Round 1 4/5.

Show that there is a unique sequence of positive integers (a_n) satisfying the following conditions: $a_1 = 1$, $a_2 = 2$, $a_4 = 12$, $a_{n+1}a_{n-1} = a_n^2 \pm 1$ for n = 2, 3, 4, …

A21.4. Estonian Mathematical Contest 1998 Selectional round

Let k a fixed positive integer. Define the sequence (E_n) by the recurrence $E_1 = k + 1$, $E_{n+1} = E_n^2 - kE_n + k$ $(n \geq 1)$. Prove that the terms of (E_n) are pairwise coprime.

A21.5. Mathematical Olympiads' Correspondence Program 1996 Canada

Suppose that n is a positive integer. Prove that there is a positive integer k for which $\left(\sqrt{2} - 1\right)^n = \sqrt{k} - \sqrt{k-1}$.

Recursions 3.

A22.1. Japanese Mathematical Olympiad 1992 First Round

The sequence of numbers $\{a_n\}$ is defined by $a_0 = 1$, $a_1 = 2$ and $a_{n+2} = a_n + (a_{n+1})^2$. What is the remainder dividing a_{1991} by 7?

A22.2. Belarusian Mathematical Competition 1995, 11^{th} grade

Determine a_{1995}, if $a_0 = a_1 = 1$ and $a_{n+2} = (n + 3)a_{n+1} – (n + 1)a_n$, for all $n \geq 0$.

A22.3. German National Mathematics Competition 1999 First Round

The sequences $a_1, a_2, a_3, \dots$, and $b_1, b_2, b_3, \dots$ are defined by $a_1 = b_1 = 1$ and $a_{n+1} = a_n + b_n$, $b_{n+1} = a_n \cdot b_n$ for $n = 1, 2, 3, \dots$ Prove that every two elements of the first sequence are relatively prime.

A22.4. Bulgarian Mathematical Competition February 2001

A sequence $a_1, a_2, \dots , a_n, \dots$ is defined by $a_1 = k$, $a_2 = 5k – 2$ and $a_{n+2} = 3a_{n+1} – 2a_n$, $n \geq 1$, where k is a real number. Prove that if k = 1 then $a_{n+2} = \left[\dfrac{7a_{n+1}^2 - 8a_n a_{n+1}}{1 + a_n + a_{n+1}}\right]$, $n \geq 1$, where [x] denotes the integer part of x.

A22.5. British Mathematical Olympiad February 2002: Round 2 (Time: 3.5 hours) 3/4.

Prove that the sequence defined by $y_0 = 1$, $y_{n+1} = \dfrac{1}{2}\left(3y_n + \sqrt{5y_n^{\,2} - 4}\right)$, $(n \geq 0)$ consists only of integers.

Polynomials

A23.1. Macedonian Mathematical Competition 2002 I Round IV Class

Factoris the polynomial $P(x) = x^4 + x^3 – 8x^2 + 3x + 5$.

A23.2. Mathematical competition in Lithuania October 2001

Find all polynomials p(x) such that the equality $p(3x)p(– 3x) = 81(x^2 – 1)^2$ holds for all real x.

A23.3. Czech and Slovak Mathematical Olympiad October 2001
(Problems for the take-home part)

Find all polynomials P(x) with real coefficients which satisfy the equality

$(x + 1)P(x – 1) + (x – 1)P(x + 1) = 2xP(x)$

for any real number x.

A23.4. Math Olympiad in Slovenia 1998 Final Round Grade Three 2/4.

Find all polynomials p with real coefficients such that $(x – 8)p(2x) = 8(x – 1)p(x)$ for every $x \in R$.

A23.5. Mathematical Olympiads' Correspondence Program 1996 Canada

Let p be an odd prime, and let $(1 + x)^{p-2} = 1 + a_1x + a_2x^2 + \dots + a_{p-2}x^{p-2}$. Show that $a_1 + 2$, $a_2 - 3$, $a_3 + 4$, …, $a_{p-3} - (p - 2)$, $a_{p-2} + (p - 1)$ are all multiples of p.

Rational Numbers

A24.1. Albanian Mathematical Olympiad March 2002 10th class 4/5.

a) Prove that $\log_{\sqrt{2}} 3$ is irrational.
b) Does there exist irrational numbers x and y such that x^y is a rational number?

A24.2. Math Olympiad in Slovenia 1998 First Round Grade Four 2/4.

Let a_1 be an arbitrary digit ($a_1 \in \{0, 1, \dots, 9\}$). For every $n \in N$, denote by a_{n+1} the last digit of the number $19a_n + 98$ in decimal representation. Prove that the number $0,a_1a_2a_3\dots$ is rational.

A24.3. Estonian Mathematical Contest 1998 Final round

Let a be a real number such that $\frac{1}{a} = a - [a]$, where [a] denotes the greatest integer not exceeding a. Prove that *a* is not a rational number.

A24.4. Austrian Mathematical Olympiad May 2001: National Competition (Day 2) 2/3.

Determine all integers m for which the solutions of the equation $3x^3 - 3x^2 + m = 0$ are all rational.

A24.5. Mathematical competition in Lithuania October 2001

Prove that on the unit circle with the centre in coordinate origin there are infinitely many points having both coordinates rational.

Trigonometry

A25.1. Macedonian Mathematical Competition 2002 I Round III Class

Evaluate the expression

$$\frac{\sin 3x}{\sin x} + \frac{\sin 6x}{\sin 2x} + \dots + \frac{\sin 3nx}{\sin nx} - \frac{\cos 3x}{\cos x} - \frac{\cos 6x}{\cos 2x} - \dots - \frac{\cos 3nx}{\cos nx}.$$

A25.2. Mathematical Competition 1997 Lithuania

Prove that, for all real x and for all positive integers n, $|\cos x| + |\cos 2x| + |\cos 4x| + \dots + |\cos 2^n x| > \frac{n}{3}$.

A25.3. Czech and Slovak Mathematical Olympiad January 2002: Second Round

Show that for any numbers $\alpha, \beta \in \left]0, \frac{\pi}{2}\right[$, the inequality $\frac{1}{\cos\alpha} + \frac{1}{\cos\beta} \geq 2\sqrt{\tan\alpha + \tan\beta}$ holds, and find the condition of equality.

A25.4. Albanian Mathematical Olympiad March 2002 10th class 5/5.

In a triangle are denoted by α, β, γ the angles in the opposite of its sides a, b, c respectively, p $= \frac{a+b+c}{2}$, and r and R are its inradius and circumradius. Prove the following relations:

a) $a\cos^2\frac{\alpha}{2} + b\cos^2\frac{\beta}{2} + c\cos^2\frac{\gamma}{2} = p\left(1+\frac{r}{R}\right)$;

b) $\sqrt{a}\cos\frac{\alpha}{2} + \sqrt{b}\cos\frac{\beta}{2} + \sqrt{c}\cos\frac{\gamma}{2} \leq \sqrt{3p\left(1+\frac{r}{R}\right)}$.

A25.5. Estonian IMO Slection Test 2002 Second Day 2/3.

Let $0 < \alpha < \frac{\pi}{2}$ and $x_1, x_2, \dots, x_n$ be real numbers such that $\sin x_1 + \sin x_2 + \dots + \sin x_n \geq n\cdot\sin\alpha$. Prove that $\sin(x_1 - \alpha) + \sin(x_2 - \alpha) + \dots + \sin(x_n - \alpha) \geq 0$.

Functions, Functional Equations 1.

A26.1. Albanian Mathematical Olympiad March 2002 11th class 1/5.

Find all the functions f: $R^* \to R$ such that $f(x) - 3f\left(\frac{1}{x}\right) = 3^x, \quad \forall x \in R^*$.

A26.2. Estonian Mathematical Olympiad 1999 Final Round 11th Grade 2/5.

Find the value of the expression

$$f\left(\frac{1}{2000}\right) + f\left(\frac{2}{2000}\right) + \dots + f\left(\frac{1999}{2000}\right) + f\left(\frac{2000}{2000}\right) + f\left(\frac{2000}{1999}\right) + \dots + f\left(\frac{2000}{1}\right)$$

assuming $f(x) = \frac{x^2}{1+x^2}$.

A26.3. South Africa, Rhodes Camp April 2001: Test 1 (Time: 4.5 hours) 1/3.

Let $\mathbf{N}^+ = \{1, 2, 3, \dots\}$ be the set of all natural numbers and f: $\mathbf{N}^+ \to \mathbf{N}^+$ be a function. Suppose $f(1) = 1$, $f(2n) = f(n)$ and $f(2n + 1) = f(2n) + 1$ for all natural numbers n.

a) Calculate the maximum value M of f(n) for $n \in \mathbf{N}^+$ with $1 \leq n \leq 2001$.

b) Find all $n \in \mathbf{N}^+$, with $1 \le n \le 2001$, such that f(n) = M.

A26.4. Thai Mathematical Olympiad 2001

Let f be a function on **N** such that f(0) = f(1) = 1 and $f(n) = \frac{-f(n-2)}{n}$ for n ≥ 2. Show that for each n, $f(2n) = \frac{(-1)^n}{2^n n!}$ and $f(2n+1) = \frac{(-1)^n 2^n n!}{(2n+1)!}$.

A26.5. Olimpíada Brasileira de Matemática 2001 Segunda Fase Nível 3 3/6.

Determine todas as funções f: R → R tais que f(x) = f(– x) e f(x + y) = f(x) + f(y) + 8xy + 115 para todos os reais x e y.

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Determine all the functions f: R → R such that for every real x, y the following holds: f(x) = f(– x) e f(x + y) = f(x) + f(y) + 8xy + 115.

Functions, Functional Equations 2.

A27.1. Olimpíada Brasileira de Matemática 2001 Primeira Fase Nível 3

Seja $f(x) = x^2 - 3x + 4$. Quantas soluções reais tem a equaçâo f(f(…(f(x)))) = 2 (onde f é aplicada 2001 vezes?

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Let $f(x) = x^2 - 3x + 4$. Determine the number of real solutions of f(f(…(f(x)))) = 2 (we applied f 2001 times).

A27.2. Math Olympiad in Slovenia 1998 Final Round Grade Four 2/4.

Find all functions f: R → R for which $f(x) + xf(1 - x) = x^2 + 1$ for every $x \in R$.

A27.3. Thai Mathematical Olympiad 2001

Assume a, b, c ∈ R. Let $f(x) = ax^4 + bx^3 + cx^2$ be such that $f[x(x + 1)] - f[x(x - 1)] = x^7$. Given $1^7 + 2^7 + 3^7 + \ldots + n^7 = \frac{n^2}{24}(n+1)^2 p(n)$ when p(n) is a function depending only on n, find p(n).

A27.4. Estonian Mathematical Contests 1996

Find all functions f: R → R satisfying the following conditions for all x ∈ R.
a) f(x) = – f(– x);
b) f(x + 1) = f(x) + 1;

c) $f\left(\frac{1}{x}\right)=\frac{1}{x^2}f(x)$, if $x \neq 0$.

A27.5. Belarusian Selection Test 1997

Find all functions $f: R \to R$ such that $f(x + y) + f(x)f(y) = f(xy) + f(x) + f(y)$.

Functions, Functional Equations 3.

A28.1. Mathematical Competition 1997 Lithuania

Is the function $f(x) = \sin x + \cos(x^2)$ periodic?

A28.2. Iranian Mathematical Olympiad 1995

Does there exist a function $f: R \to R$ that fulfils all of the following conditions:
a) $f(1) = 1$,
b) there exists $M > 0$ such that $-M < f(x) < M$,
c) if $x \neq 0$ then $f\left(x+\frac{1}{x^2}\right)=f(x)+\left(f\left(\frac{1}{x}\right)\right)^2$?

A28.3. Irish Mathematical Olympiad 1999: Second Day (Time: 3 hours) 2/5.

A function $f: N \to N$ (where N denotes the set of positive integers) satisfies
1) $f(ab) = f(a)f(b)$ whenever the greatest common divisor of a and b is 1;
2) $f(p + q) = f(p) + f(q)$ for all prime numbers p and q.
Prove that $f(2) = 2$, $f(3) = 3$ and $f(1999) = 1999$.

A28.4. Japanese Mathematical Olympiad 2002 First Round

Let Q the set of rational numbers. Consider a function $f: Q \to Q$ satisfying the following conditions 1, 2, 3:

1) $f(0) = 2$, $f(1) = 3$.
2) For every rational number x, and for all integers n, $f(x + n) - f(x) = n(f(x + 1) - f(x))$.
3) For each nonzero rational number x, $f(x) = f\left(\frac{1}{x}\right)$.

Find the rational solutions of $f(x) = 2002$.

A28.5. Romanian selection examination for IMO 1997

Find the functions $f: R \to [0, \infty)$ such that $f(x^2 + y^2) = f(x^2 - y^2) + f(2xy)$, for all real numbers x and y.

Analysis

A29.1. Mathematical national competition in Croatia 2002: 4th grade 1/4.

Determine the sum of the series

$s = 1 + 4x + 9x^2 + \dots + n^2x^{n-1} + \dots,$

where $|x| < 1$.

A29.2. Mathematical Competition 1992 New Zealand

The following formulae define the sequences u_n and v_n for n = 1, 2, …:

$u_1 = 0$, $u_{n+1} = \frac{1}{2}(u_n + v_n)$; $v_1 = 1$, $v_{n+1} = \frac{1}{4}(u_n + 3v_n)$.

a) Find the first four terms of each sequence.

b) Show that u_n approaches $\frac{2}{3}$ as n approaches infinity.

c) What is the limit of the sequence v_n ?

A29.3. Bulgarian Mathematical Competition February 2001

A sequence $a_1, a_2, \dots, a_n, \dots$ is defined by $a_1 = k$, $a_2 = 5k - 2$ and $a_{n+2} = 3a_{n+1} - 2a_n$, $n \ge 1$, where k is a real number. Find those values of k, for which that the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent.

A29.4. Japanese Mathematical Olympiad 2002 First Round

Find the minimum value of the following: $x + y + \frac{2}{x+y} + \frac{1}{2xy}$ $(x, y > 0)$.

A29.5. Estonian Mathematical Olympiad 1999 Final Round 12th Grade 2/5.

Find the value of the integral $\int_{-1}^{1} \ln\left(x + \sqrt{1 + x^2}\right) dx$.